An Abstraction of Evolution through Algebra

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Abstract

Evolution algebras were created in order to model non-Mendelian genetics in an algebraic fashion. In this application, elements of the evolution algebra represent distributions of some hereditary unit, such as an allele, throughout a population. Multiplication of these elements represents some sort of non-Mendelian reproduction, such as asexual reproduction. Evolution algebras also have many interesting mathematical properties. They are generally nonassociative algebras, but do not fall into any of the well-known categories of nonassociative algebras such as Jordan algebras or Lie algebras. Furthermore, they possess a natural correspondence with a wide range of other fields in mathematics. In this thesis, I develop a generalization of evolution algebras. I then explore the correspondence of evolution algebras with Markov chains, graph theory, and formal grammar theory.

Introduction

0.1 Biological Motivation

Although this thesis deals with evolution on a very abstract level, it is helpful to have at least a vague notion of some of the motivating genetics. So, assuming you remember nothing from seventh grade biology, I present a short crash course in genetics. A unit of hereditary information is called a **gene**. For instance, there are genes corresponding to eye color, hair color, height, etc. Just as eye color and hair color vary, so do the underlying genes that code for these traits. A variation of a gene is called an **allele**. That is, in the genes that code for eye color, one allele corresponds to blue while another to brown and another to green. Genes are all clumped together on **chromosomes**. The chromosomes carry all the genetic information of an organism. **Diploid** organisms carry two sets of each chromosome. That is, both chromosomes in a pair typically carry all the same genes, but the alleles may differ. It is the combination of these alleles that determines the phenotypic trait corresponding to the gene, e.g. eye color or blood type. Some cells carry only one copy of each chromosome. These cells are called **haploid**. Some organsisms carry only haploid cells. In diploid organisms, the sex cells, called **gametes**, are typically haploid cells. During fertilization process (be it sexual or otherwise), the gamete cells of the male and female organisms merge, creating a **zygote**, which is a diploid cell (one set of chromosomes coming from the female, the other set coming from the male).

Genetic algebras concerning diploid organisms have been actively studied over the past century. In fact, Mendel used a symbolic notation very suggestive of algebra while he developed his laws. Mathematicians soon picked up on the algebraic nature of Mendel's laws and created many full blown algebras to model Mendelian inheritance. The basic idea behind these algebras (and behind the algebras in this thesis) is that each element of the algebra represents a distribution of alleles throughout a population. Multiplying two elements represents the organisms in the one element reproducing with the organisms in the other element. Thus, multiplication can be used to track the distribution of alleles throughout time. Note that evolution algebras, despite the name, have very little to do with natural selection. The algebra essentially assumes that all the organisms have equal fitness and then asks how alleles will spread. I believe that these algebras could be augmented to deal with natural selection, but that is not the goal of this thesis and would likely be extraordinarily difficult.

Although original attempts were mostly concerned with using algebra as a tool for biology, it was soon found that these algebras had very interesting mathematical properties. In particular, these algebras are not associative, but also do not typically fall into any of the well known categories of nonassociative algebras, such as Lie or Jordan algebras. An excellent overview of Mendelian algebras can be found in Reed [1997].

Not all of inheritance is governed by Mendel's laws, however. For example, prokaryotes (such as bacteria), which constitued some of the earliest forms of life on Earth, reproduce by binary fission. That is, when a prokaryote reproduces, the organism's chromosome is copied and then the cell simply splits in half, one half containing one copy of the chromosome, the other containing the other half. So called non-Mendelian inheritance is not usually studied with these algebras. To repair this gap, Jianjun Paul Tian developed the theory of **evolution algebras**. Although these evolution algebras seem targeted towards a rather specific sort of reproduction, as you will see, they are very good at capturing discrete dynamic behavior in general, much in the same way Markov chains do.

0.2 Mathematical Motivation

Evolution algebras have a natural correspondence to many branches of mathematics. In this thesis, I explore their connection with Markov chains, graph theory, and formal grammar theory. The purpose of the expositions of these three correspondences is not to show anything ground-breaking, but to demonstrate that evolution algebras provide a natural way to algebraize these different fields of mathematics. Evolution algebras provide a unified, algebraic language to graph theory, Markov chains, formal grammar theory, as well as other fields not discussed here. Furthermore, as each of these fields have been widely studied and are very well understood, they can in turn inform evolution algebra theory, to help it overcome the difficulties the nonassociativity creates.

The mathematical applications of evolution algebras are certainly not limited to the ones I present in this thesis. For instance, Tian describes how one might pursue a connection between evolution algebras and topology. In an email, he informed me that he currently has an undergraduate student developing a correspondence between evolution algebras and braid theory.

0.3 Chapter Outline

In the first chapter, I define an algebra that can describe both Mendelian and non-Mendelian inheritance in general, called a generalized gametic algebra. Gametic algebras as I present them are a generalization of the gametic algebras developed in Reed [1997]. Many of the basic properties of evolution algebras hold for gametic algebras as well, so I begin with gametic algebras to maintain as much generality as possible. I then define evolution algebras as a particular kind of gametic algebra, and demonstrate what further properties evolution algebras have. Again, I work with a generalization of the evolution algebras developed in Tian [2007]. One of the most appealing aspects of evolution algebras is their ability to model dynamical systems. After establishing their basic properties, I investigate a mathematical analogue of descendance. One of the primary concerns of evolution algebra theory is determining under what conditions we can say one entity will descend from another, and what implications for the structure of the algebra as a whole this has. Finally, I develop one of the most structurally interesting aspects of evolution algebras: their hierarchical organization.

In the subsequent chapters, I examine the relationship between evolution algebras and other fields of mathematics. I spend the second chapter looking at the correspondence between Markov chains and evolution algebras. This is perhaps the most obvious related field of mathematics as it makes explicit the underlying dynamics of evolution algebras. In the third chapter, we see a very rich correspondence with graph theory. Many of the important properties of graphs and evolution algebras relate turn out to be nearly identical. In particular, I demonstrate how evolution algebras can be used to solve a simple graph theory problem, suggesting how the fields may benefit each other. Finally, I use formal grammar theory to derive decidability results concerning descendants.

Chapter 1

Evolution Algebras for Their Own Sake

1.1 Gametic Algebras

1.1.1 Definition and Basic Properties

Definition 1.1.1. Let $\mathcal{S}$ be a nonassociative polynomial ring with the set of algebra generators E over a commutative ring R with identity. E may either be finite or countably infinite, with elements enumerated as $\{e_1, e_2, e_3 \ldots\}$. Let I be the smallest ideal of $\mathcal{S}$ containing:

$$\{e_i e_j - \sum_{e_k \in E} a_{ijk} e_k | e_i, e_j \in E, i < j, a_{ijk} \in R, \text{ only finite many } a_{ijk} \text{ are nonzero}\}$$

Let $\Gamma\langle R, E, I\rangle$ be the R-submodule of $\mathcal{S}/I$ generated by E and closed under the multiplication operation inherited from $\mathcal{S}$. We call $\Gamma\langle R, E, I\rangle$ a **generalized gametic algebra**. Then each element of $\Gamma\langle R, E, I\rangle$ can be written as a polynomial in the e_i with coefficients in R and the constant term being zero.

Remark 1.1.2. These gametic algebras are generalized in the sense that both Reed [1997] and Tian [2007] require R to be the real numbers. I will usually just refer to generalized gametic algebras as gametic algebras.

Notation 1.1.3. I will use $\sum^{\not\infty}$ to signify that a possibly infinite sum has nonzero coefficients for only finitely many of its summands.

Theorem 1.1.4. *Let $\Gamma\langle R, E, I\rangle$ be a gametic algebra. Then any element of $\Gamma\langle R, E, I\rangle$ may be written as a (finite) linear combination of the elements in E.*

Proof. Every element of $\Gamma\langle R, E, I\rangle$ can be written as a polynomial over the elements of E with coefficients in R, where the constant coefficient is zero. Hence, it suffices to show that for any $x \in \Gamma\langle R, E, I\rangle$ that can be written as a term, x can be rewritten as a linear combination of the elements in E. To this end, I will use induction over the degree of x written as a term.

Let $x \in \Gamma\langle R, E, I\rangle$ be a term of degree 1. That is, $x = ce_i$ for some $c \in R$ and $e_i \in E$. Thus, x is a linear combination of elements in E.

So let's suppose that $\deg x = n+1$ and that all terms of degree less than or equal to n are linear combinations of elements in E. In this case, x can be written as (cyz), where $y, z \in \Gamma\langle R, E, I\rangle$ are also terms and $c \in R$. By definition, $\deg x = \deg y + \deg z$ and so $1 \leq \deg y \leq n$ and $1 \leq \deg z \leq n$. Thus, we have $y = \sum_{e_i \in E}^{\not\infty} a_i e_i$ and $z = \sum_{e_i \in E}^{\not\infty} b_i e_i$, where $a_i, b_i \in R$. Then:

$$\begin{aligned} x &= cyz \\ &= c\Big(\sum_{e_i \in E}^{\not\infty} a_i e_i\Big)\Big(\sum_{e_i \in E}^{\not\infty} b_i e_i\Big) \\ &= c\Big(\sum_{e_i \in E}^{\not\infty} a_i e_i\Big(\sum_{e_j \in E}^{\not\infty} b_j e_j\Big)\Big) \\ &= c\Big(\sum_{e_i, e_j \in E}^{\not\infty} a_i b_j e_i e_j\Big) \end{aligned}$$

Thus, we have shown x to be a linear combination of quadratics. Each quadratic is by assumption equal to a linear combination of elements in E, and therefore so is x.

Hence, by induction, we find that all terms, and therefore all elements in $\Gamma\langle R, E, I\rangle$ are linear combinations of elements in E. □

Corollary 1.1.5. *Using the notation from Definition 1.1.1, $\Omega\langle R, E, I\rangle$ just is the R-submodule of $\mathcal{S}/I$ generated by E with an induced multiplication operation. That is, closure under multiplication adds no additional elements.*

Theorem 1.1.6. *Let $\Gamma\langle R, E, I\rangle$ be a gametic algebra. Then:*
1) Multiplication distributes over addition in general.
2) Multiplication is not associative in general.

Proof. 1. Let $x, y, z \in \Gamma\langle R, E, I\rangle$, so that

$$x = \sum_{e_i \in E}^{\not\infty} a_i e_i, \quad y = \sum_{e_i \in E}^{\not\infty} b_i e_i, \quad z = \sum_{e_i \in E}^{\not\infty} c_i e_i$$

where $a_i, b_i, c_i \in R$. Then we have (note: we know scalar multiplication distributes over polynomial addition already):

$$\begin{aligned} x(y+z) &= \Big(\sum_{e_i \in E}^{\not\infty} a_i e_i\Big)\Big(\sum_{e_i \in E}^{\not\infty} b_i e_i + c_i e_i\Big) \\ &= \Big(\sum_{e_i \in E}^{\not\infty} a_i e_i\Big)\Big(\sum_{e_i \in E}^{\not\infty} (b_i + c_i) e_i\Big) \\ &= \sum_{e_i, e_j \in E, i \leq j}^{\not\infty} a_j (b_i + c_i) e_i e_j \\ &= \sum_{e_i, e_j \in E, i \leq j}^{\not\infty} (a_j b_i e_i e_j + a_j c_i e_i e_j) \\ &= xy + xz \end{aligned}$$

Thus, multiplication distributes over addition in general.

2. Although the nonassociative polynomial ring from which $\Gamma\langle R, E, I\rangle$ is derived is, well, nonassociative, we have to make sure that the relations imposed by I do not force associativity. To this end, consider the following example:

   Let $R = \mathbb{Z}$, $E = \{e_1, e_2\}$, and I be generated by $\{e_1e_1 - e_2, e_1e_2 - 0, e_2e_2 - e_1\}$. That is, in $\Gamma\langle R, E, I\rangle$:

$$e_1 e_1 = e_2 \qquad e_1 e_2 = 0 \qquad e_2 e_2 = e_1$$

   So, then, testing associativity, we find:

$$\begin{aligned} e_1(e_2e_2) &= e_1e_1 \\ &= e_1 \\ (e_1e_2)e_2 &= 0e_2 \\ &= 0 \end{aligned}$$

   Thus, this gametic algebra is not associative. □

Remark 1.1.7. Now that we have some of the most basic properties of gametic algebras established, we need to guarantee that the generators of these algebras are free. That is, we need to make sure that, for any $\Gamma\langle R, E, I\rangle$, if $\sum_{e_i \in E}^{\not\infty} a_i e_i = 0$ with $a_i \in R$, then $a_i = 0$ for all i. I do this in Theorem 1.1.8. But first, to motivate this a bit, note that if gametic algebras were associative, generators would not be free in general. Consider the following example:

Let A be a gametic algebra, with generators $E = \{e_1, e_2\}$ and relations I generated by:

$$\{e_1e_1 - 2e_1, e_1e_2 + e_1 - e_2, e_2e_2 - 0\}$$

That is, in A:

$$e_1e_1 = 2e_1 \qquad e_1e_2 = -e_1 + e_2 \qquad e_2e_2 = 0$$

Now, suppose we add to A all the necessary relations so that associativity holds in general. Then we have the following:

$$\begin{aligned} 0 &= 0 + 0 \\ &= (1 + e_2)0 + (1 - e_1)0 \\ &= (1 + e_2)(e_1^2 - 2e_1) + (1 - e_1)(e_1e_2 + e_1 - e_2) \\ &= e_1^2 - 2e_1 + e_1^2e_2 - 2e_1e_2 + e_1e_2 + e_1 - e_2 - e_1^2e_2 - e_1^2 + e_1e_2 \\ &= -e_1 - e_2 \end{aligned}$$

Therefore, E is not free in A in this case.

However, we should note that, for some gametic algebras, associativity will hold just by the nature of the relations. For instance, let $\Gamma\langle \mathbb{Z}, E, I\rangle$ be a gametic algebra, where $E = \{e_1, e_2\}$ and I contains:

$$\{e_1e_1 - e_1, e_1e_2 - e_2, e_2e_2 - e_2\}$$

Then, we have:

$$\begin{aligned} (e_1e_1)e_2 &= e_1e_2 \\ &= e_1(e_1e_2) \\ e_1(e_2e_2) &= e_1e_2 \end{aligned}$$

and so we see that $\Gamma\langle \mathbb{Z}, E, I\rangle$ is associative. This implies that general associativity brings in many more relations than the simple "quadratic equals linear" relations of gametic algebras.

Theorem 1.1.8. *Let $\Gamma\langle R, E, I\rangle$ be a gametic algebra constructed from a nonassociative polynomial ring $\mathcal{S}$ as in Definition 1.1.1. Then E is a free set in $\Gamma\langle R, E, I\rangle$. That is, $\sum_{e_l \in E} a_l e_l = 0$ if and only if $a_l = 0$ for all l.*

Proof. Suppose there exists $\{a_1, \dots, a_n, \dots\} \subseteq R$ such that $\sum\limits_{e_l \in E}^{\not\infty} a_l e_l = 0 \in \Gamma\langle R, E, I\rangle$ and at least one a_l is not zero. Consider $\sum\limits_{e_l \in E}^{\not\infty} a_l e_l$ as an element of $\mathcal{S}$. Then we have:

$$\sum_{e_l \in E}^{\not\infty} a_l e_l = \sum_{e_i, e_j \in E}^{\not\infty} d_{ij}\left(e_i e_j - \sum_{e_k \in E}^{\not\infty} a_{ijk} e_k\right)$$

where $d_{ij} \in \mathcal{S}$. For notational convenience, let $A_{ij} = (e_i e_j - \sum\limits_{e_k \in E}^{\not\infty} a_{ijk} e_k)$.

We order the quadratic monomials of $\mathcal{S}$ as follows: $e_i e_j < e_{i'} e_{j'}$ for $i \le j$ and $i' \le j'$ if and only if $i \le i'$ and $i = i' \Rightarrow j < j'$. That is, $e_1 e_1 < e_1 e_2 < \dots < e_2 e_2 < e_2 e_3 \dots$ (for convenience, we will think of the ordered pairs (i, j) as being ordered in this way as well).

We rewrite each d_{ij} as follows: Consider the smallest (i, j) such that $d_{ij} \neq 0$. Suppose d_{ij} has a quadratic term $c e_{i'} e_{j'}$ (where $c \in R$) such that $e_{i'} e_{j'} > e_i e_j$. In this case, we rewrite the term $c e_{i'} e_{j'} A_{ij}$ as follows:

$$\begin{aligned}
c e_{i'} e_{j'} A_{ij} &= c e_{i'} e_{j'} A_{ij} - (c \sum_{e_k \in E}^{\not\infty} a_{i'j'k} e_k) A_{ij} + (c \sum_{e_k \in E}^{\not\infty} a_{i'j'k} e_k) A_{ij} \\
&= (c(e_{i'} e_{j'} - \sum_{e_k \in E}^{\not\infty} a_{i'j'k} e_k)) A_{ij} + (c \sum_{e_k \in E}^{\not\infty} a_{i'j'k} e_k) A_{ij} \\
&= (c A_{i'j'}) A_{ij} + (c \sum_{e_k \in E}^{\not\infty} a_{i'j'k} e_k) A_{ij} \\
&= (c A_{ij}) A_{i'j'} + (c \sum_{e_k \in E}^{\not\infty} a_{i'j'k} e_k) A_{ij}
\end{aligned}$$

We then move the cA_{ij} into $d_{i'j'}$ and the $(c \sum\limits_{e_k \in E}^{\not\infty} a_{i'j'k} e_k)$ into d_{ij}. Notice that there is no longer a $e_{i'}e_{j'}$ term in d_{ij}. Repeat this process for all other $e_{i'}e_{j'}$ for which $e_{i'}e_{j'} > e_ie_j$. Thus, d_{ij} contains no quadratic terms whose monomial is larger than e_ie_j. Repeat this process for the next largest (i,j) and so forth. Thus, for all (i,j), d_{ij} contains no quadratic terms whose monomial is larger than e_ie_j.

Let (i,j) be a pair of indices such that $d_{ij} \neq 0$. Consider the contents of $d_{ij}A_{ij}$:

$$d_{ij} = (\text{const. term } + \text{ lin. terms } + \text{ quad. terms } + \text{ lin.} \cdot \text{quad. terms} + \ldots)$$
$$A_{ij} = (e_ie_j - \sum_{e_k \in E}^{\not\infty} a_{ijk}e_k)$$

Now, I will show that no d_{ij} may have a quadratic term. Let $c \in R$ be the coefficient of e_pe_q in d_{ij} (c might be zero). Then, once we multiply out $d_{ij}A_{ij}$, we will have a $c(e_pe_q)(e_ie_j)$ term. This term must cancel with something or $c = 0$. Note that $(p,q) \leq (i,j)$. For all $(i',j') \neq (i,j)$, any *quad.* $\cdot$ *quad.* term in $d_{i'j'}A_{i'j'}$ has as a factor $(e_{i'}e_{j'})$. So $c(e_pe_q)(e_ie_j)$ can cancel with no terms found in $d_{i'j'}A_{i'j'}$. Thus, $c = 0$. Therefore, no d_{ij} may contain a quadratic term.

Next, I will show that d_{ij} can only have a constant term. Let y be a monomial of degree 1 or degree greater than 2, and let $c \in R$ such that cy appears in d_{ij}. Then, once we multiply out $d_{ij}A_{ij}$, we will have a $c(y)(e_ie_j)$ term. Note that since $\mathcal{S}$ is nonassociative, the only (nonconstant) factors of this term are y and e_ie_j. Again, this term must cancel or $c = 0$. However, since no d_{pq} has a quadratic term, the only $d_{pq}A_{pq}$ that has some term with a factor of (e_ie_j) and has a degree greater than 2 can be $d_{ij}A_{ij}$. Thus, $c = 0$. Therefore, no d_{ij} may have terms of degree 1 or more. That is, all d_{ij} are elements of R.

Finally, again consider d_{ij}. Then $d_{ij}A_{ij}$ has a term $d_{ij}(e_ie_j)$, which is a quadratic term since $d_{ij} \in R$. But of course, the only quadratic term that any $d_{i'j'}A_{i'j'}$ for $(i',j') \neq (i,j)$ may have is $d_{i'j'}(e_{i'}e_{j'})$. Since d_{ij} and $d_{i'j'}$ are constants, no term may cancel with $d_{ij}(e_ie_j)$, and therefore $d_{ij} = 0$. Thus, all $d_{ij} = 0$. Hence, we have in $\mathcal{S}$:

$$\begin{aligned}\sum_{e_l \in E}^{\not\infty} a_le_l &= \sum_{e_i,e_j \in E}^{\not\infty} d_{ij}\left(e_ie_j - \sum_{e_k \in E}^{\not\infty} a_{ijk}e_k\right)\\ &= \sum_{e_i,e_j}^{\not\infty} 0 \cdot \left(e_ie_j - \sum_{e_k \in E}^{\not\infty} a_{ijk}e_k\right)\\ &= 0\end{aligned}$$

The only way this can hold is if all $a_l = 0$ (there is no question that E is free in $\mathcal{S}$).

When we again pass down to $\Gamma\langle R, E, I\rangle$, all a_l must still be 0. Thus, E is a free set in $\Gamma\langle R, E, I\rangle$.

□

Corollary 1.1.9. *Let $\Gamma\langle R, E, I\rangle$ be a gametic algebra. Then $\Gamma\langle R, E, I\rangle$ is a free module and E is a basis for $\Gamma\langle R, E, I\rangle$.*

Proof. This is an immediate consequence of Theorem 1.1.4 and Theorem 1.1.8. □

Corollary 1.1.10. *Let $\Gamma\langle R, E, I\rangle$ and $\Gamma\langle R, F, J\rangle$ be gametic algebras. Then their direct sum, $\Gamma\langle R, E, I\rangle \oplus \Gamma\langle R, F, J\rangle$, is a gametic algebra as well.*

Proof. $\Gamma\langle R, E, I\rangle \oplus \Gamma\langle R, F, J\rangle$ has a generating set

$$G = \{(e_i, 0), (0, f_j) | e_i \in E, f_j \ \in \ F\}$$

It suffices to show that multiplying two elements in G results in a linear combination of elements in G:

$$\begin{aligned}
(e_i, 0)(e_j, 0) &= (e_i e_j, 0) \\
&= (\sum_{e_k \in E}^{\not\infty} a_{ijk} e_k, 0) \\
&= \sum_{e_k \in E}^{\not\infty} a_{ijk}(e_k, 0) \\
(0, f_i)(0, f_j) &= (0, f_i f_j) \\
&= (0, \sum_{f_k \in E}^{\not\infty} b_{ijk} f_k) \\
&= \sum_{f_k \in F}^{\not\infty} b_{ijk}(0, f_k) \\
(e_i, 0)(0, f_j) &= (0, 0)
\end{aligned}$$

Thus, $\Gamma\langle R, E, I\rangle \oplus \Gamma\langle R, F, J\rangle$ is a gametic algebra. □

1.1.2 Powers

Definition 1.1.11. Let G be a gametic algebra. Then we define the **principal powers** of $g \in G$ as follows:

$$g^n = \begin{cases} g & \text{if } n = 1 \\ (g^{n-1})g & \text{if } n > 1 \end{cases}$$

We define the **plenary powers** of $g \in G$ as follows:

$$g^{[n]} = \begin{cases} g & \text{if } n = 0 \\ (g^{[n-1]})(g^{[n-1]}) & \text{if } n > 0 \end{cases}$$

Corollary 1.1.12. *Let G be a gametic algebra and $g \in G$. Then $(g^{[n]})^{[m]} = g^{[m+n]}$ for all $m \in \mathbb{N}$ and $n \in \mathbb{Z}^+$.*

Proof. To this end, we will use induction over m. First, suppose $m = 0$. Then we have:

$$\begin{aligned} (g^{[n]})^{[m]} &= (g^{[n]})^{[0]} \\ &= g^{[n]} \\ &= g^{[n+0]} \\ &= g^{[n+m]} \end{aligned}$$

Now, suppose that $(g^{[n]})^{[k]} = g^{[k+n]}$ holds for all $0 \leq k \leq m$. I must show that it holds for $m + 1$ as well:

$$\begin{aligned} (g^{[n]})^{[m+1]} &= (g^{[n]})^{[m]}(g^{[n]})^{[m]} \\ &= (g^{[n+m]})(g^{[n+m]}) \\ &= g^{[n+m+1]} \end{aligned}$$

Therefore, $(g^{[n]})^{[m]} = g^{[m+n]}$ for all $m \in \mathbb{N}$ and $n \in \mathbb{Z}^+$. □

1.1.3 Norms and Genetic Realization

Definition 1.1.13. Let $\Gamma\langle R, E, I\rangle$ be a gametic algebra and R an ordered ring with a norm function $|\cdot|$. Then we define a module norm fuction, $|\cdot|$, for $\Gamma\langle R, E, I\rangle$ as follows:

$$|\sum_{e_i \in E}^{\not\infty} a_i e_i| = \sum_{e_i \in E}^{\not\infty} |a_i|$$

Verifying that this function is in fact a norm is simple:

1. *Nonnegativity:*

$$0 \leq \sum_{e_i \in E}^{\not\infty} |a_i| = |\sum_{e_i \in E}^{\not\infty} a_i e_i|$$

   with equality only if each a_i is 0. In this case, $\displaystyle\sum_{e_i \in E}^{\not\infty} a_i e_i = 0$.

2. *Linearity:* Let $a, b \in R$ and $e_i \in E$. Then

$$|abe_i| = |ab| = |a||b| = |a||be_i|$$

   Thus, for any $x \in \Gamma\langle R, E, I\rangle$ and $a \in R$, $|ax| = |a||x|$.

3. *Triangle Inequality:*

$$\begin{aligned} |\sum_{e_i \in E}^{\not\infty} a_i e_i + \sum_{e_i \in E}^{\not\infty} b_i e_i| &= |\sum_{e_i \in E}^{\not\infty} (a_i + b_i) e_i| \\ &= \sum_{e_i \in E}^{\not\infty} |a_i + b_i| \\ &\leq \sum_{e_i \in E}^{\not\infty} (|a_i| + |b_i|) \\ &= |\sum_{e_i \in E}^{\not\infty} a_i e_i| + |\sum_{e_i \in E}^{\not\infty} b_i e_i| \end{aligned}$$

Definition 1.1.14. Let $\Gamma\langle \mathbb{R}, E, I\rangle$ be a gametic algebra such that for all $e_i, e_j \in E$:

$$|e_i e_j| = |\sum_{e_k \in E}^{\not\infty} a_{ijk} e_K| = 1$$

and each $a_{ijk} \geq 0$. Then $\Gamma\langle \mathbb{R}, E, I\rangle$ is said to have **genetic realization**.

1.2 Evolution Algebras

The generality and nonassociativity of gametic algebras makes it rather difficult to prove anything about them. Indeed, the title of this thesis makes no mention of gametic anything. From here on out, we will examine a specialized kind of gametic algebra called an evolution algebra.

1.2.1 Definition and Basic Properties

Definition 1.2.1. An **evolution algebra** $\Omega\langle R, E, I\rangle$ is a gametic algebra with the following restriction on multiplication (for $e_i, e_j \in E$ and $a_{ik} \in R$):

$$e_i e_j = 0 \qquad \text{for } i \neq j$$

Furthermore, we call E the **natural basis** of $\Omega\langle R, E, I\rangle$, where a natural basis is a basis of $\Omega\langle R, E, I\rangle$ that follows the given restriction on multiplication.

Remark 1.2.2. Often, I will refer to E as *the* natural basis of $\Omega\langle R, E, I\rangle$. Although $\Omega\langle R, E, I\rangle$ may have other natural bases, we single out E for pragmatic reasons. The original motivation behind evolution algebras was to model the changing frequencies of alleles in a population of organisms. The elements of the generating set represent the alleles of interest, whereas linear combinations of the generators represent some statistical distribution of them. Similarly, in the following chapters I will demonstrate a close relationship between evolution algebras and Markov chains, graphs, and formal grammars. Each of these correspondences relies on singling out E. Although focusing solely on the intrinsic properties of evolution algebras (and thus not singling out E) may prove interesting, I will not explore that path in this thesis.

Corollary 1.2.3. *Let $\Omega\langle R, E, I\rangle$ and $\Omega\langle R, F, J\rangle$ be evolution algebras. Then their direct sum, $\Omega\langle R, E, I\rangle \oplus \Omega\langle R, F, J\rangle$, is an evolution algebra as well.*

Proof. $\Omega\langle R, E, I\rangle \oplus \Omega\langle R, F, J\rangle$ has the basis

$$G = \{(e_i, 0), (0, f_j) | e_i \in E, f_j \in F\}$$

We know that gametic algebras are closed under direct sum, and so we only have to show that multiplying distinct elements in G is zero:

$$\begin{aligned}
(e_i, 0)(e_j, 0) &= (e_i e_j, 0) && i \neq j \\
&= (0, 0) \\
(0, f_i)(0, f_j) &= (0, f_i f_j) \\
&= (0, 0)
\end{aligned}$$

Therefore, G can serve as the natural basis of $\Omega\langle R, E, I\rangle \oplus \Omega\langle R, F, J\rangle$. Thus, evolution algebras are closed under direct sum. □

Definition 1.2.4.

1. An evolution algebra A is called **trivial** if $A = \{0\}$. In general, we will never work with the trivial evolution algebra.

2. Let $A = \Omega\langle R, E, I\rangle$ be an evolution algebra and A_0 a submodule of A. If A_0 has a natural basis $E_0 \subseteq E$, then we call A_0 an **evolution subalgebra**.

3. Let A be a nontrivial evolution algebra and A_0 an evolution subalgebra. If $A_0 \neq A$ then we call A_0 a **proper evolution subalgebra**.

4. For the **direct sum** of two evolution algebras, $\Omega\langle R, E, I\rangle$ and $\Omega\langle R, F, J\rangle$ are distinct evolution subalgebras of $\Omega\langle R, E, I\rangle \oplus \Omega\langle R, F, J\rangle$ since we can simply identify $e_i \in E$ with $(e_i, 0)$ and $f_i \in F$ with $(0, f_i)$.

5. An nontrivial evolution algebra is **connected** if it cannot be decomposed into a direct sum of proper evolution subalgebras.

6. An nontrivial evolution algebra is **irreducible** if it has no proper evolution subalgebras.

7. Let A be an evolution algebra and A_0 an evolution subalgebra of A. If $A_0 A \subseteq A_0$ then A_0 is an **evolution ideal**.

8. A nontrivial evolution algebra is **simple** if it has no proper evolution ideal.

9. Let $A = \Omega\langle R, E, I\rangle$ be an evolution algebra where R is an ordered ring. Then A is a **nonnegative evolution algebra** iff $\forall e_i, e_j \in E$ where $e_i e_j = \sum_{e_k \in E}^{\not\infty} a_{ijk} e_k$, $a_{ijk} \geq 0$. In this case, let $A^+ = \{\sum_{e_i \in E}^{\not\infty} a_i e_i \in A | a_i > 0\}$.

10. Let $\Omega\langle R, E, I\rangle$ and $\Omega\langle S, F, J\rangle$ be two evolution algebras such that there exists a mapping $\phi : \Omega\langle R, E, I\rangle \to \Omega\langle S, F, J\rangle$ for which $\phi(E) \subseteq F$ and for any $x, y \in \ \Omega\langle R, E, I\rangle$:

$$\phi(xy) = \phi(x)\phi(y) \qquad \phi(x + y) = \phi(x) + \phi(y)$$

Then $\Omega\langle R, E, I\rangle$ and $\Omega\langle S, F, J\rangle$ are **homomorphic evolution algebras** and ϕ is an **evolution homomorphism**. If ϕ is bijective, then it is an **evolution isomorphism**.

Note that I require $\phi(E) \subseteq F$ so that for any evolution subalgebra A_0 of $\Omega\langle R, E, I\rangle$, $\phi(A_0)$ is an evolution subalgebra of $\Omega\langle S, F, J\rangle$.

Remark 1.2.5. Again, notice that the definition of an evolution subalgebra given here is not intrinsic and hence neither is the definition of an evolution isomorphism. This demonstrates the extent to which E in $\Omega\langle R, E, I\rangle$ is considered special. To see the significance of this, consider the following example:

Let $\Omega\langle \mathbb{Z}, E, I\rangle$ be an evolution algebra such that $E = \{e_1, e_2, e_3\}$ and I imposes the multiplication rules:

$$e_1^2 = e_1 + e_2 + e_3 \qquad e_2^2 = e_2 + e_3 \qquad e_3^2 = e_2 + e_3$$

Now, the ideal generated by e_1 (in the typical algebraic sense) contains:

$$\begin{aligned} &e_1 \\ e_1^2 &= e_1 + e_2 + e_3 \\ e_2(e_1^2) &= e_2 + e_3 = e_3(e_1^2) \\ (e_1 + e_2 + e_3)^2 &= e_1 + 3(e_2 + e_3) \end{aligned}$$

Thus, this ideal does have *a* natural basis, $\{e_1, e_2 + e_3\}$, the sense that $e_1(e_2 + e_3) = 0$. But $\{e_1, e_2 + e_3\}$ is not a subset of E and so cannot be *the* natural basis of the ideal. This ideal has no natural basis that is a subset of E and so it is not an evolution ideal.

Theorem 1.2.6. *Let $\Omega\langle R, E, I\rangle$ be an evolution algebra and $A \subseteq \Omega\langle R, E, I\rangle$. Then A is an evolution subalgebra if and only if A is an evolution ideal.*

Proof. By definition, if A is an evolution ideal than A is an evolution subalgebra. Thus, I only need to show that if A is an evolution subalgebra, then A is an evolution ideal. So suppose that A is an evolution subalgebra. Then A has a basis $E_1 \subseteq E$. Let $x = \sum_{e_i \in E_1}^{\not\infty} x_i e_i \in A$ and $y = \sum_{e_i \in E}^{\not\infty} y_i e_i \in \Omega\langle R, E, I\rangle$, where $x_i, y_i \in R$. Then we have:

$$\begin{aligned} xy &= (\sum_{e_i \in E_1}^{\not\infty} x_i e_i)(\sum_{e_i \in E}^{\not\infty} y_i e_i) \\ &= \sum_{e_i \in E_1}^{\not\infty} \sum_{e_j \in E}^{\not\infty} x_i y_i e_i e_j \\ &= \sum_{e_i \in E_1}^{\not\infty} x_i y_i e_i^2 \end{aligned}$$

since when $i \neq j$, $e_i e_j = 0$. Of course, $\sum_{e_i \in E_1}^{\not\infty} x_i y_i e_i^2 \in A$, and thus $A\Omega\langle R, E, I\rangle \subseteq A$. Therefore, A is an evolution ideal. □

Corollary 1.2.7. *The following are immediate results of the previous theorem:*

1. *An evolution algebra is simple if and only if it is irreducible.*
2. *A simple evolution algebra is connected.*

Theorem 1.2.8. *Let $\Omega\langle R, E, I\rangle$ be an evolution algebra and let $\Omega\langle R, E_0, I\rangle$ and $\Omega\langle R, E_1, I\rangle$ be evolution subalgebras (so $E_0, E_1 \subseteq E$). Then $\Omega\langle R, E_0, I\rangle \cap \Omega\langle R, E_1, I\rangle$ is an evolution subalgebra of $\Omega\langle R, E, I\rangle$.*

Proof. Note that $\Omega\langle R, E_0 \cap E_1, I\rangle$ is an evolution subalgebra of both $\Omega\langle R, E_0, I\rangle$ and $\Omega\langle R, E_1, I\rangle$. Thus we have:

$$\Omega\langle R, E_0 \cap E_1, I\rangle \subseteq \Omega\langle R, E_0, I\rangle \cap \Omega\langle R, E_1, I\rangle$$

Now, let $x \in \Omega\langle R, E_0, I\rangle \cap \Omega\langle R, E_1, I\rangle$. Then x must be a linear combination of the elements of E_0 and, separately, a linear combination of the elements of E_1. Since the elements of E are linearly independent, x must then be a linear combination of the elements of $E_0 \cap E_1$. Hence:

$$\Omega\langle R, E_0, I\rangle \cap \Omega\langle R, E_1, I\rangle \subseteq \Omega\langle R, E_0 \cap E_1, I\rangle$$

Therefore, $\Omega\langle R, E_0, I\rangle \cap \Omega\langle R, E_1, I\rangle = \Omega\langle R, E_0 \cap E_1, I\rangle$ is an evolution subalgebra. □

Notation 1.2.9. Let $A \subseteq \Omega\langle R, E, I\rangle$. Then $\langle A\rangle$ is the smallest evolution subalgebra of $\Omega\langle R, E, I\rangle$ that contains A. The previous theorem guarantees that $\langle A\rangle$ is well-defined. Similarly, for $x \in \Omega\langle R, E, I\rangle$, I will simply write $\langle x\rangle$.

Corollary 1.2.10. *Let $\Omega\langle R, E, I\rangle$ be an evolution algebra and $e_i \in E$. Then $\langle e_i\rangle$ is connected.*

Proof. Suppose that $\langle e_i\rangle = G_1 \oplus G_2$ (where G_1 and G_2 are evolution subalgebras of $\Omega\langle R, E, I\rangle$). Then $G_1 \cap G_2 = \{0\}$. So, without loss of generality, assume $e_i \in G_1$. But then $G_2 = \{0\}$ since $\langle e_i\rangle$ is the smallest evolution algebra containing e_i. □

Proposition 1.2.11. *Evolution algebras do not have maximal evolution subalgebras in general. That is, for an evolution algebra $\Omega\langle R, E, I\rangle$, there may not exist a proper evolution subalgebra which is contained in no other proper evolution subalgebra.*

Proof. Let $\Omega\langle R, E, I\rangle$ be defined as follows. Let $E = \{e_1, e_2, e_3, \dots\}$. We define multiplication so that:

$$e_i^2 = \sum_{e_j \in E}^{\infty} a_j e_j \qquad a_j = \begin{cases} 0 & \text{if } j > i \\ 1 & \text{if } j \leq i \end{cases}$$

Thus, the multplication table looks like:

$$\begin{aligned} e_1^2 &= e_1 \\ e_2^2 &= e_1 + e_2 \\ e_3^2 &= e_1 + e_2 + e_3 \\ &\vdots \end{aligned}$$

Let $E_0 \subseteq E$ be finite. Let $e_i \in E_0$ be such that, for any $e_j \in E_0$, $i \geq j$. Then we have

$$\langle E_0\rangle = \langle e_i\rangle \subsetneq \langle e_{i+1}\rangle \subsetneq \Omega\langle R, E, I\rangle$$

Thus, no proper evolution subalgebra with a finite natural basis can be maximal.

Now, let $E_1 \subseteq E$ be infinite. Then, for any $e_i \in E$, there exists some $e_j \in E_0$ for which $j \geq i$. Thus, $e_i \in \langle E_1\rangle$. Therefore, $\langle E_1\rangle = \Omega\langle R, E, I\rangle$. Hence, there do not exist any proper evolution subalgebras with an infinite natural basis.

So we find that $\Omega\langle R, E, I\rangle$ has no maximal evolution subalgebra. □

1.2.2 Evolution Operators

Definition 1.2.12. Let $\Omega\langle R, E, I\rangle$ be an evolution algebra. Let the L be the R-linear mapping taking $\Omega\langle R, E, I\rangle$ to $\Omega\langle R, E, I\rangle$, induced by:

$$L(e_i) = e_i^2 \qquad e_i \in E$$

Then L is the **evolution operator** for $\Omega\langle R, E, I\rangle$. For an arbitrary element in $\Omega\langle R, E, I\rangle$, we have:

$$L(\sum_{e_i \in E}^{\not\infty} a_i e_i) = \sum_{e_i \in E}^{\not\infty} a_i L(e_i) = \sum_{e_i \in E}^{\not\infty} a_i e_i^2 \qquad a_i \in R$$

by the R-linearity of L.

Consider $\Omega\langle R, E, I\rangle$ as a free module. Since L maps each basis vector e_i to $e_i^2 = \sum_{e_k \in E}^{\not\infty} a_{ki} e_k$, we can represent L as follows:

$$L = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & \dots \\ a_{21} & a_{22} & \dots & a_{2n} & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Note that this matrix will have dimension equal to the size of E. Hence, it may be finite or countably infinite.

Theorem 1.2.13. *Let $\Omega\langle R, E_0, I\rangle \subseteq \Omega\langle R, E, I\rangle$ be an evolution subalgebra and L the evolution operator of $\Omega\langle R, E, I\rangle$. Then the image of L restricted to $\Omega\langle R, E_0, I\rangle$ is a subset of $\Omega\langle R, E_0, I\rangle$. In other words, L restricted to $\Omega\langle R, E_0, I\rangle$ is an evolution operator for that evolution subalgebra.*

Proof. We have:

$$L(\sum_{e_i \in E_0}^{\not\infty} a_i e_i) = \sum_{e_i \in E_0}^{\not\infty} a_i e_i^2$$

By definition, each $e_i^2 \in \Omega\langle R, E_0, I\rangle$, and therefore $\sum_{e_i \in E_0}^{\not\infty} a_i e_i^2 \in \Omega\langle R, E_0, I\rangle$. □

1.3 Dynamics and Hierarchies of Evolution Algebras

We are now ready to begin investigating the properties of evolution algebras as models of dynamic systems.

1.3.1 Occurrence Relations

Notation 1.3.1. Let $\rho_i : \Omega\langle R, E, I\rangle \to \Gamma\langle R, E, I\rangle$ be the **projection operator**, defined as follows. Let $\sum\limits_{e_j \in E}^{\not\infty} a_j e_j$ be an arbitrary element in $\Omega\langle R, E, I\rangle$. Then:

$$\rho_i(\sum_{e_j \in E}^{\not\infty} a_j e_j) = a_i e_i.$$

Similarly, let $\rho_i^o : \Omega\langle R, E, I\rangle \to \Omega\langle R, E, I\rangle$ be the **deletion operator**, defined:

$$\rho_i^o(\sum_{e_j \in E}^{\not\infty} a_j e_j) = \sum_{e_j \in E, j \neq i}^{\not\infty} a_j e_j$$

Note that both ρ_i and ρ_i^o are linear.

Definition 1.3.2. Let $x \in \Omega\langle R, E, I\rangle$ and $e_i \in E$. Then e_i is said to **occur in** x iff $\rho_i(x) \neq 0$. We denote this $e_i \prec x$. For any $e_i, e_j \in E$, if there exists some $n \in \mathbb{N}$ such that $e_i \prec e_j^{[n]}$, then we say that e_i is a **descendant** of e_j. Note that e_i is always a descendent of itself, since $e_i^{[0]} = e_i$.

Let $e, f \in E$ and suppose we can form a sequence of elements of E

$$e = e_0, e_1, \ldots, e_{n-1}, e_n = f$$

such that each e_i is a descendant of e_{i+1}. Then we call that sequence a **sequence of descendants** beginning with e and ending with f. Let $E_0 \subseteq E$ such that for any $f \in E_0$ and $e \in E$, if there exists a sequence of descendants beginning with e and ending with f, then $e \in E_0$. Then we say that E_0 is **closed under descendants**.

Remark 1.3.3. Note that the descendant relation is not transitive in general. For example, let $\Omega\langle R, E, I\rangle$ be an evolution algebra and $E = \{e_1, e_2, e_3, e_4\}$, with multiplication defined:

$$e_1^2 = e_2 + e_3 \qquad e_2^2 = e_4$$
$$e_3^2 = -e_4 \qquad e_4^2 = e_4$$

Hence, we have:

$$\begin{aligned} e_1^{[2]} &= (e_2 + e_3)^{[1]} \\ &= e_4 - e_4 \\ &= 0 \end{aligned}$$

Thus, e_4 is not a descendant of e_1, though e_4 is a descendant of both e_2 and e_3. Hence, this invites the need for the concept of a sequence of descendants. Note that in this example, both e_4, e_3, e_1 and e_4, e_2, e_1 are sequences of descendants beginning with e_4 and ending with e_1.

For non-negative evolution algebras, however, the descendant relation is transitive as we will see in Theorem 1.3.8 below.

Corollary 1.3.4. *Let $e_i, e_j \in E$ with $\Omega\langle R, E, I\rangle$. If e_i is a descendant of e_j then $\langle e_i\rangle \subseteq \langle e_j\rangle$.*

Proof. Because e_j is a descendant of e_i, it must occur in $e^{[n]}$ for some n. Now, $\langle e_j\rangle$ must have a natural basis $E_j \subseteq E$. Since E is a free set in $\Omega\langle R, E, I\rangle$, $e_i \in E_j$, and therefore, $\langle e_i\rangle \subseteq \langle e_j\rangle$. Note that this corollary would not work if we did not make pick out E to be *the* natural basis of $\Omega\langle R, E, I\rangle$. □

Theorem 1.3.5. *Let $e \in E$ and $E_0 \subseteq E$ for $\Omega\langle R, E, I\rangle$. Then $e \in \langle E_0\rangle$ if and only if there there exists some $f \in E_0$ and some sequence of generators $e = e_1, \ldots, e_{n-1}, e_n = f$ such that each e_k is a descendant of e_{k+1}. Note that this is a generalization of the previous corollary.*

Proof. To this end, I will walk through the construction of $\langle E_0\rangle \cap E$ (all generators that appear in $\langle E_0\rangle$). Let E_0^k denote the following:

$$E_0^k = \begin{cases} E_0 & \text{if } k = 1 \\ \{e_i : \exists e_j \in E_0^{k-1}, e_i \prec e_j^2\} & \text{if } k > 1 \end{cases}$$

Thus, E_0^k is all the generators which appear in the square of any generator E_0^{k-1}. To construct $\langle E_0 \rangle \cap E$, we simply take the union of all E_0^k:

$$\langle E_0 \rangle \cap E = \cup_{k=1}^{\infty} E_0^k$$

To see this, note that by construction, the linear combination of the elements in $\cup_{k=1}^{\infty} E_0^k$ will be closed under multiplication. Furthermore, $e \in \cup_{k=1}^{\infty} E_0^k$ if and only if there exists a sequence of descendants starting with e and ending with some $f \in E_0$. □

Corollary 1.3.6. *Let $\Omega\langle R, E, I\rangle$ be an evolution algebra and $E_0 \subseteq E$. Then E_0 is closed under descendants if and only if $\langle E_0 \rangle \cap E = E_0$.*

Proof. This is immediate from the previous theorem. □

Lemma 1.3.7. *Let $H = \Omega\langle R, E, I\rangle$ be a nonnegative evolution algebra. Let $x, y \in H^+$ and $n \in \mathbb{N}$. Then $\exists z \in H^+$ such that $(x+y)^{[n]} = x^{[n]} + z$.*

Proof. To this end, we will use induction over n. So let $n = 0$. Then we have:

$$\begin{aligned} (x+y)^{[n]} &= (x+y)^{[0]} \\ &= x + y \\ &= x^{[n]} + y \end{aligned}$$

Thus, the lemma holds for $n = 0$.

Now, assume that the lemma holds for some $n > 1$. Let $w \in H^+$ be such that $(x+y)^{[n]} = x^{[n]} + w$. Then we have:

$$\begin{aligned} (x+y)^{[n+1]} &= (x^{[n]} + w)^{[1]} \\ &= x^{[n]}x^{[n]} + 2x^{[n]}w + ww \\ &= x^{[n+1]} + 2x^{[n]}w + w^{[1]} \end{aligned}$$

So we see that $(x+y)^{[n+1]} = x^{[n+1]} + z$, where $z = 2x^{[n]}w + w^{[1]}$. Thus, the lemma holds for $n+1$ as well. Therefore, by induction, the lemma holds for all $n \in \mathbb{N}$. □

Theorem 1.3.8. *Let $H = \Omega\langle R, E, I\rangle$ be a nonnegative evolution algebra and let $e_i, e_j, e_k \in E$. Let $m, n \in \mathbb{N}$ be such that $e_i \prec e_j^{[n]}$ and $e_j \prec e_k^{[m]}$. Then $e_i \prec e_k^{[n+m]}$.*

Proof. We know that $e_k^{[m]} = a_j e_j + x$ for some $a_j > 0$ and $x \in H^+$. By the previous lemma, we have:

$$\begin{aligned} e_k^{[n+m]} &= (e_k^{[m]})^{[n]} \\ &= (a_j e_j + x)^{[n]} \\ &= (a_j e_j)^{[n]} + y && \text{for some } y \in H^+ \\ &= a_j^{(2^n)} e_j^{[n]} + y \\ &= a_j^{(2^n)} (b_i e_i + z) + y \end{aligned}$$

for some $b_j > 0$ and $z \in H^+$. Note that e_i will not cancel with anything in y or z since we are in a nonnegative evolution algebra. Thus, $e_i \prec e_k^{[n+m]}$. □

Corollary 1.3.9. *Let $\Omega\langle R, E, I\rangle$ be a nonnegative evolution algebra and $e_i, e_j \in E$. Then $\langle e_i\rangle \subseteq \langle e_j\rangle$ if and only if e_i is a descendant of e_j.*

Proof. By Corollary 1.3.4, we know that if e_i is a descendent of e_j, then $\langle e_i\rangle \subseteq \langle e_j\rangle$. So assume $\langle e_i\rangle \subseteq \langle e_j\rangle$. By Theorem 1.3.5, there must be a sequence of descendants beginning with e_i and ending with e_j. However, by Theorem 1.3.8, the descendant relation is transitive in $\Omega\langle R, E, I\rangle$. Thus, e_i is a descendant of e_j. □

Definition 1.3.10. Let $\Omega\langle R, E, I\rangle$ be an evolution algebra and $e_i, e_j \in E$. If e_i and e_j are descendants of each other then we say e_i and e_j **intercommunicate**.

Theorem 1.3.11. *Let $\Omega\langle R, E, I\rangle$ be a nonnegative evolution algebra. Let the relation $\leq$ be defined as follows: for any $e_i, e_j \in E$, $e_i \leq e_j$ if and only if either $e_i = e_j$ or e_i is a descendant of e_j but e_j is not a descendant of e_i. Then E is a partially ordered set under $\leq$.*

Proof.

1. *Reflexivity:* Reflexivity is built into the definition.
2. *Antisymmetry:* Let $e_i, e_j \in E$ such that $e_i \leq e_j$ and $e_j \leq e_i$. Then e_i and e_j must be descendants of each other. But, by definition, this can only happen if $e_i = e_j$.
3. *Transitivity:* Let $e_i, e_j, e_k \in E$ such that $e_i \leq e_j$ and $e_j \leq e_k$. Then, there exists m and n such that $e_i \prec e_j^{[m]}$ and $e_j \prec e_k^{[n]}$. Then, by the above proposition, $e_i \prec e_k^{[m+n]}$. Thus, e_i is a descendant of e_k. Furthermore, e_k cannot be a descendant of e_i since that would imply that e_j is a descendant of e_i. Therefore, $e_i \leq e_k$.

Hence, $\leq$ enforces a partial ordering on E. □

1.3.2 Algebraic Persistency and Algebraic Transiency

Definition 1.3.12. Let $\Omega\langle R, E, I\rangle$ be an evolution algebra and $e \in E$. We call e **algebraically persistent** (or just persistent) if $\langle e\rangle$ is a simple subalgebra. If e is not persistent, then we call e **algebraically transient** (or just transient). That is, a generator e is algebraically transient if and only if $\langle e\rangle$ has a proper subalgebra.

Theorem 1.3.13. *Let $\Omega\langle R, E, I\rangle$ be an evolution algebra. Then each $e \in E$ is persistent if and only if $\Omega\langle R, E, I\rangle$ is the direct sum of one or more simple evolution subalgebras.*

Proof. Suppose each $e \in E$ is persistent. First I claim that for any $e_i, e_j \in E$, $\langle e_i\rangle$ and $\langle e_j\rangle$ are either identical or $\langle e_i\rangle \cap \langle e_j\rangle = \{0\}$. Since both $\langle e_i\rangle$ and $\langle e_j\rangle$ are simple, $\langle e_i\rangle \cap \langle e_j\rangle$ cannot be a proper evolution subalgebra of either. Hence, it is $\{0\}$, or $\langle e_i\rangle = \langle e_j\rangle$. Thus, we may then partition E up into subsets, $E_0, E_1, E_2, \ldots$, such that $e_i, e_j \in E_k$ if and only if $\langle e_i\rangle = \langle e_j\rangle$. Since every $e \in E$ is in one of these subsets and since $\langle E_i\rangle \cap \langle E_j\rangle = \{0\}$, we have

$$\Omega\langle R, E, I\rangle = \langle E_0\rangle \oplus \langle E_1\rangle \oplus \langle E_2\rangle \oplus \cdots$$

where each $\langle E_k\rangle$ is simple.

Now, suppose $\Omega\langle R, E, I\rangle$ is the direct sum of one or more simple evolution subalgebras:

$$\Omega\langle R, E, I\rangle = \Omega\langle R, E_0, I\rangle \oplus \Omega\langle R, E_1, I\rangle \oplus \Omega\langle R, E_2, I\rangle \oplus \cdots$$

Then, by definition, for any $e \in E_k$, e is persistent. Therefore, all $e \in E$ are persistent. □

Corollary 1.3.14.

1. *Let $\Omega\langle R, E, I\rangle$ be connected. Then $\Omega\langle R, E, I\rangle$ has a proper evolution subalgebra if and only if $\Omega\langle R, E, I\rangle$ has an algebraically transient generator.*
2. *The descendants of an algebraically persistent generator are persistent as well.*

Proof. Both of these are direct results of the previous theorem. □

Theorem 1.3.15. *Let $\Omega\langle R, E, I\rangle$ be finite dimensional. Then $\Omega\langle R, E, I\rangle$ has a simple evolution subalgebra.*

Proof. Let $n = |E|$. If $\Omega\langle R, E, I\rangle$ has no proper evolution subalgebras, then we are done. So suppose $\Omega\langle R, E, I\rangle$ has a proper subaglebra $\Omega\langle R, E_0, I\rangle$. Then E_0 is a proper subset of E. If $\Omega\langle R, E_0, I\rangle$ is simple, then we are done, so assume it is not. Let $\Omega\langle R, E_1, I\rangle$ be a proper subalgebra of $\Omega\langle R, E_0, I\rangle$, and thus E_1 is a proper subset of E_0. Again, if $\Omega\langle R, E_1, I\rangle$ is simple, we are done, so assume it is not. In this way, we can create a decreasing chain of proper evolution subalgebras. Since E is finite, this chain must terminate. The last proper subalgebra on the chain will thus be simple. □

1.3.3 Semidirect-sum Decomposition of Evolution Algebras

Notation 1.3.16. Let $\Omega\langle R, E, I\rangle$ be an evolution algebra and A_0 and A_1 be distinct submodules (recall that $\Omega\langle R, E, I\rangle$ is a free module). Then let $A_0 \dot{+} A_1$ denote the **semidirect submodule sum** of A_0 and A_1. That is, $A_0 \dot{+} A_1$ is the smallest *submodule* of $\Omega\langle R, E, I\rangle$ containing A_0 and A_1. In general, $A_0 \dot{+} A_1$ is not be an evolution algebra.

Definition 1.3.17. Let $\Omega\langle R, E, I\rangle$ be an evolution algebra and $E_0 \subseteq E$ the set of all its algebraically transient generators. Let B be the submodule of $\Omega\langle R, E, I\rangle$ spanned by E_0. We call B the **transient space** of $\Omega\langle R, E, I\rangle$.

Theorem 1.3.18. *Let $\Omega\langle R, E, I\rangle$ be connected. Let $A = \{A_0, A_1, A_2, \dots\}$ be the set of all simple evolution subalgebras of $\Omega\langle R, E, I\rangle$ and let B be the transient space of $\Omega\langle R, E, I\rangle$. Then:*

$$\Omega\langle R, E, I\rangle = (A_0 \oplus A_1 \oplus A_2 \oplus \dots) \dot{+} B$$

Proof. Since each A_i is simple, $A_i \cap A_j = \{0\}$ for all i and j. Furthermore, since no A_i can contain an algebraically transient generator, $B \cap A_i = \{0\}$ for all i. Let $E_i \subseteq E$ be the natural basis of A_i and T the set of transient generators of $\Omega\langle R, E, I\rangle$ (a basis of B). So we have:

$$(E_1 \cup E_2 \cup \cdots) \cup T = E$$
$$(E_1 \cap E_2 \cap \cdots) \cap T = \emptyset$$

Thus, $(A_0 \oplus A_1 \oplus A_2 \oplus \dots) \dot{+} B$ is the free module spanned by $(E_1 \cap E_2 \cap \dots) \cap T$, which is $\Omega\langle R, E, I\rangle$. □

Let B_n be the transient space of some evolution algebra $\Omega\langle R, E, I\rangle$. The purpose of the index n on B_n will become clear in the next subsection. For now, I write it so that we can get used to the notation. Although B_n is not an evolution algebra, the operations of $\Omega\langle R, E, I\rangle$ do impose some kind of algebraic structure on it. We can turn B_n into an evolution algebra using the following induced multiplication:

Let $E_n \subseteq E$ be the free module basis of B_n. We define the multiplication operation $\overset{n}{\cdot}$ on B_n:

$$e_i \overset{n}{\cdot} e_j = 0 \qquad \text{if } i \neq j$$
$$e_i \overset{n}{\cdot} e_i = \rho_{B_n}(e_i e_j)$$

That is, multiplication of B_n is just like multiplication of $\Omega\langle R, E, I\rangle$, but with all terms containing persistent generators removed. Thus, we see that transient spaces can easily be turned into evolution algebras.

1.3.4 Hierarchy of an Evolution Algebra

Let $\Omega\langle R, E, I\rangle$ be an evolution algebra. We can use semidirect-sum decomposition to reveal a hierarchical structure within $\Omega\langle R, E, I\rangle$. We define each level of the hierarchy through the following recursion:

- Let $A_{0,0}, A_{0,1}, \ldots$ be the simple subalgebras of $\Omega\langle R, E, I\rangle$ and B_0 its transient space, called the 0**th transient space**. The 0**th structure** of $\Omega\langle R, E, I\rangle$ is the decomposition of $\Omega\langle R, E, I\rangle$, given by:

$$\Omega\langle R, E, I\rangle = (A_{0,0} \oplus A_{0,1} \oplus \ldots) \dot{+} B_0$$

- Consider B_0 as an evolution algebra in the sense as at the end of the previous section. Let $A_{1,0}, A_{1,1}, \ldots$ be the simple subalgebras of B_0 and B_1 its transient space, called the 1**st transient space**. The 1**st structure** of $\Omega\langle R, E, I\rangle$ is the decomposition of B_0, given by:

$$B_0 = (A_{1,0} \oplus A_{1,1} \oplus \ldots) \dot{+} B_1$$

- Consider B_{n-1} as an evolution algebra. Let $A_{n,0}, A_{n,1}, \ldots$ be the simple subalgebras of B_{n-1} and B_n its transient space, called the n**th transient space**. The n**th structure** of $\Omega\langle R, E, I\rangle$ is the decomposition of B_{n-1}, given by:

$$B_{n-1} = (A_{n,0} \oplus A_{n,1} \oplus \ldots) \dot{+} B_n$$

Definition 1.3.19. I will call each of the $A_{n,j}$ in the above characterization of the hierarchical structure of an evolution algebra **hierarchically simple evolution subalgebras**, meaning that each $A_{n,j}$ is a simple evolution subalgebra at some nth structure in the hierarchy, given the adopted operator $\overset{n-1}{\cdot}$.

Corollary 1.3.20. *Let $\Omega\langle R, E, I\rangle$ be finite dimensional and nontrivial.*

1. *For each n, the nth structure of $\Omega\langle R, E, I\rangle$ consists of a direct sum of only finitely many simple evolution subalgebras, in addition to a semidirect-sum with the nth transient space.*

2. *There exists some m such that the mth transient space of $\Omega\langle R, E, I\rangle$ is $\{0\}$. That is, the hierarchical structure of $\Omega\langle R, E, I\rangle$ has only finitely many levels.*

Proof. Both of these results follow immediately from the fact that simple evolution subalgebras must be nontrivial. □

Lemma 1.3.21. *Let $\Omega\langle R, E, I\rangle$ be an evolution algebra. Let $e_i, e_j \in E$ such that e_j is a descendant of e_i and they both appear in the $(m+1)$th structure of $\Omega\langle R, E, I\rangle$. Then, in the evolution algebra B_m (the mth transient space), e_j is still a descendant of e_i under the modified operator $\overset{m}{\cdot}$.*

Proof. To this end, I will use induction over m. So let $m = 0$. Let $k \in \mathbb{N}$ be so that $e_j \prec e_i^{[k]}$ in $\Omega\langle R, E, I\rangle$. I claim that $e_j \prec e_i^{[k]}$ in B_0 as well. For this, I will use induction over k.

Suppose $k = 0$. Then $e_i = e_j$. So let k be arbitrary and suppose that for any generator e_l which appears in B_0, if $e_l \prec e_i^{[k]}$ in $\Omega\langle R, E, I\rangle$, then $e_l \prec e_i^{[k]}$ in B_0. I must show, then, that this implies that $e_j \prec e_i^{[k+1]}$ in B_0. Since $e_j \prec e_i^{[k+1]}$ in $\Omega\langle R, E, I\rangle$, there must be some particular e_l such that $e_l \prec e_i^{[k]}$ and $e_j \prec e_l^{[1]}$. Hence, if $e_l \in B_0$, then e_j will still be a descendant of e_i in B_0. So suppose, for the sake of contradiction, that e_l is not in B_0. Then e_l must be persistent in $\Omega\langle R, E, I\rangle$. Then e_j would be persistent in $\Omega\langle R, E, I\rangle$ as well. But then $e_j \notin B_0$, which is a contradiction. Therefore, $e_l \in B_0$ and so e_j is a descendant of e_i in B_0.

Now, suppose that e_j is a descendant of e_i in B_m. Then I must show that if $e_i, e_j \in B_{m+1}$, e_j is a descendant of e_i in B_{m+1} as well. But notice that B_{m+1} is the 0th transient space of B_m. Hence, the argument just given for showing that e_j is a descendant of e_i in B_0 holds here as well. That is, B_m takes the place of $\Omega\langle R, E, I\rangle$ and B_{m+1} takes the place of B_0.

Therefore, in general, if e_j is a descendant of e_i in $\Omega\langle R, E, I\rangle$ and they both appear in $(m+1)$th structure of $\Omega\langle R, E, I\rangle$, then e_j is still a descendant of e_i in B_m under the modified operator $\overset{m}{\cdot}$. □

Lemma 1.3.22. *Let $\Omega\langle R, E, I\rangle$ be an evolution algebra. Let $e_i, e_j \in E$ such that e_i is persistent in the mth structure of $\Omega\langle R, E, I\rangle$ and e_j is persistent in the nth structure. If e_j is a descendant of e_i, then $n \leq m$.*

Proof. Suppose $n > m$. Then e_j appears in the mth structure of $\Omega\langle R, E, I\rangle$. We know that e_i must be persistent in the mth structure and e_j is a descendant of e_i. Since the descendants of a persistent generator are persistent, e_j is persistent in the mth structure as well, by the previous lemma. But then e_j will not be in the mth transient space of $\Omega\langle R, E, I\rangle$, and so will not appear in any structure higher than the mth. This is a contradiction, since e_j must appear in the nth structure of $\Omega\langle R, E, I\rangle$. Therefore, $n \leq m$. □

Corollary 1.3.23. *Let $\Omega\langle R, E, I\rangle$ be an evolution algebra.*

1. *Let $e_i, e_j \in E$ such that e_i is persistent in the mth structure of $\Omega\langle R, E, I\rangle$ and e_j is persistent in the nth structure. If $e_j \in \langle e_i\rangle$, then $n \leq m$. Furthermore, for all $k \leq n$, $e_j \in \langle e_i\rangle$ in the kth structures under the modified operator $\overset{k-1}{\cdot}$.*

2. *Let $e_i, e_j \in \Omega\langle R, E, I\rangle$. Then e_i and e_j are in the same hierarchically simple evolution subalgebra if and only if $\langle e_i\rangle = \langle e_j\rangle$.*

Note that this is a generalization of the previous two lemmas.

Proof. (1) follows from the previous two lemmas and Theorem 1.3.5. (2) follows immediately from (1). □

Definition 1.3.24. Let $\Omega\langle R, E, I\rangle$ be an evolution algebra and let A be the set of all hierarchically simple evolution subalgebras of $\Omega\langle R, E, I\rangle$. Let the hierarchically simple evolution subalgebras be indexed $A_{m,i}$, where m is structure in which $A_{m,i}$ is simple and i is its index within the mth structure. Let $D_{m,i} \subseteq A$ be defined

$$D_{m,i} = \{A_{n,j} \in A : A_{n,j} \neq A_{m,i}, (\exists e \in A_{n,j})(\exists f \in A_{m,i}) e \prec f^2\}$$

That is, $D_{m,i}$ is the set of all hierarchically simple evolution subalgebras that contain an immediate descendant of some generator in $A_{m,i}$. Let $\mathbb{S}\langle R, E, I\rangle = \Omega\langle \mathbb{Z}, A, J\rangle$; that is, the natural basis of $\mathbb{S}\langle R, E, I\rangle$ are the hierarchically simple evolution subalgebras of $\Omega\langle R, E, I\rangle$. We define J such that $\mathbb{S}\langle R, E, I\rangle$ has the following multiplication table.

$$A_{m,i}^2 = \begin{cases} \displaystyle\sum_{A_{n,j} \in D_{m,i}}^{\infty} A_{n,j} & \text{if } m \geq 1 \\ A_{m,i} & \text{if } m = 0 \end{cases}$$

We call $\mathbb{S}\langle R, E, I\rangle$ the **skeleton** of $\Omega\langle R, E, I\rangle$. Note that skeletons are nonnegative evolution algebras and therefore $\langle A_{n,j}\rangle \subseteq \langle A_{m,i}\rangle$ if and only $A_{n,j}$ is a descendant of $A_{m,i}$.

Theorem 1.3.25. *Let $\Omega\langle R, E, I\rangle$ be an evolution algebra and $\mathbb{S}\langle R, E, I\rangle = \Omega\langle \mathbb{Z}, A, J\rangle$ its skeleton. Let $e, f \in E$ such that $f \in A_{m,k}$ and $e \in A_{n,l}$, where $A_{m,k}, A_{n,l} \in A$. Then $\langle e\rangle \subseteq \langle f\rangle$ in $\Omega\langle R, E, I\rangle$ if and only if $A_{n,l}$ is a descendant of $A_{m,k}$.*

Proof. Suppose $\langle e\rangle \subseteq \langle f\rangle$ in $\Omega\langle R, E, I\rangle$. Then there exists a sequence of descendants beginning with e and ending with f. Therefore, there must exist a sequence of generators

$$e = e_1, e_2, \ldots, e_q = f$$

such that for each i, $e_i \prec e_{i+1}^2$. Therefore, the hierarchically simple evolution algebra containing e_i will either also contain e_{i+1}, or it will occur in the square of the hierarchically simple evolution subalgebra containing e_{i+1}. Since $\mathbb{S}\langle R, E, I\rangle$ is a nonnegative evolution algebra, the descendant relation is transitive, and therefore $A_{n,l}$ is a descendant of $A_{m,k}$.

Suppose that $A_{n,l}$ is a descendant of $A_{m,k}$. Then there is some $p \in \mathbb{N}$ such that $A_{n,l} \prec A_{m,k}^{[p]}$. I will perform induction over p. So let $p = 0$. Then $A_{m,k} = A_{n,l}$. Hence, by Corollary 1.3.23, $\langle e\rangle = \langle f\rangle$.

Now suppose that for some p and for any $A_{n,l}, A_{m,k} \in A$, if $A_{n,l} \prec A_{m,k}^{[p]}$, then for any $e \in A_{n,l}$ and $f \in A_{m,k}$, $\langle e\rangle \subseteq \langle f\rangle$.

Fix $A_{n,l}, A_{m,k} \in A$ such that $A_{n,l} \prec A_{m,k}^{[p+1]}$. Then there exists some $A_{n',l'} \prec A_{m,k}^{[p]}$ such that $A_{n,l} \prec A_{n',l'}^{[1]}$. Then for any generators $e \in A_{n,l}$ and $f \in A_{n',l'}$, $\langle e\rangle \subseteq \langle f\rangle$. Likewise, by assumption, for any generators $e \in A_{n',l'}$ and $f \in A_{m,k}$, $\langle e\rangle \subseteq \langle f\rangle$. Therefore, for any $f \in A_{m,k}$ and any $e \in A_{n,l}$, $\langle e\rangle \subseteq \langle f\rangle$. □

Corollary 1.3.26. *If $A_{n,l}$ is a descendant of $A_{m,k}$, then $n \leq m$, with equality if and only if $A_{n,l} = A_{m,k}$.*

Proof. This is an immediate consequence of the previous theorem. □

Lemma 1.3.27. *Let $\mathbb{S}\langle R, E, I\rangle = \Omega\langle \mathbb{Z}, A, J\rangle$ be the skeleton of some evolution algebra. For any $A_{m,k}, A_{n,l} \in A$, $\langle A_{m,k}\rangle = \langle A_{n,l}\rangle$ if and only if $A_{m,k} = A_{n,l}$.*

Proof. Of course, if $A_{m,k} = A_{n,l}$, then $\langle A_{m,k}\rangle = \langle A_{n,l}\rangle$.

So suppose $\langle A_{m,k}\rangle = \langle A_{n,l}\rangle$. Since $\mathbb{S}\langle R, E, I\rangle$ is nonnegative, $A_{m,k}$ and $A_{n,l}$ are descendants of each other. By the previous corollary, then $A_{m,k} = A_{n,l}$. □

Theorem 1.3.28. *Let $\mathbb{S}\langle R, E, I\rangle = \Omega\langle \mathbb{Z}, A, J\rangle$ be the skeleton of some evolution algebra. Then $\mathbb{S}\langle R, E, I\rangle$ is isomorphic with its skeleton, $\mathbb{S}\langle \mathbb{Z}, A, J\rangle$.*

Proof. By the previous lemma and Corollary 1.3.23, the hierarchically simple evolution subalgebras of $\mathbb{S}\langle R, E, I\rangle = \Omega\langle \mathbb{Z}, A, J\rangle$ must contain exactly one element of A. That is, there is a bijection from A to the hierarchically simple evolution subalgebras of $\mathbb{S}\langle R, E, I\rangle$. Furthermore, by Theorem 1.3.25 and its corollary, if $A_{m,i} \in A$ is simple in the mth structure of $\Omega\langle R, E, I\rangle$, then the hierarchically simple evolution subalgebra of $\mathbb{S}\langle R, E, I\rangle$ containing $A_{m,i}$ must be simple in the mth structure of $\mathbb{S}\langle R, E, I\rangle$. Hence, I label the hierarchically simple evolution subalgebra containing $A_{m,i}$ as $A'_{m,i}$.

I must show that, for any $A_{m,i}, A_{n,j} \in A$,

$$A_{n,j} \prec A_{m,i}^2 \Leftrightarrow A'_{n,j} \prec A'^2_{m,i}$$

But this is immediate from the definition of skeleton. Therefore, $\mathbb{S}\langle R, E, I\rangle$ is isomorphic with $\mathbb{S}\langle \mathbb{Z}, A, J\rangle$. That is, the skeleton of a skeleton is itself. □

Example. To better understand the notions of the hierarchy and skeleton of an evolution algebra, consider the following example. Let $\Omega\langle \mathbb{Z}, E, I\rangle$, with

$$E = \{e_1, e_2, \ldots, e_9\}$$

and let I give the following multiplication table:

$$e_1^2 = e_3 + e_4 + e_5 + e_8 \qquad e_2^2 = e_5 + e_7$$
$$e_3^2 = e_3 + e_8 \qquad e_4^2 = e_5^2$$
$$e_6^2 = e_6 + e_7 + e_9 \qquad e_7^2 = e_6 + e_7$$
$$e_8^2 = e_8 \qquad e_9^2 = e_9$$

This give the following decomposition. Note that $\langle \cdot \rangle_m$ indicates that the evolution subalgebra uses the modified operator of the mth transient space, $\overset{m}{\cdot}$.

$$\begin{aligned}
&\text{0th structure:} && \Omega\langle R,E,I\rangle = (\langle e_8\rangle \oplus \langle e_9\rangle) \dot{+} B_0 \\
&\text{1st structure:} && B_0 = (\langle e_3\rangle_0 \oplus \langle e_4, e_5\rangle_0 \oplus \langle e_6, e_7\rangle_0) \dot{+} B_1 \\
&\text{2nd structure:} && B_1 = \langle e_1\rangle_1 \oplus \langle e_2\rangle_1
\end{aligned}$$

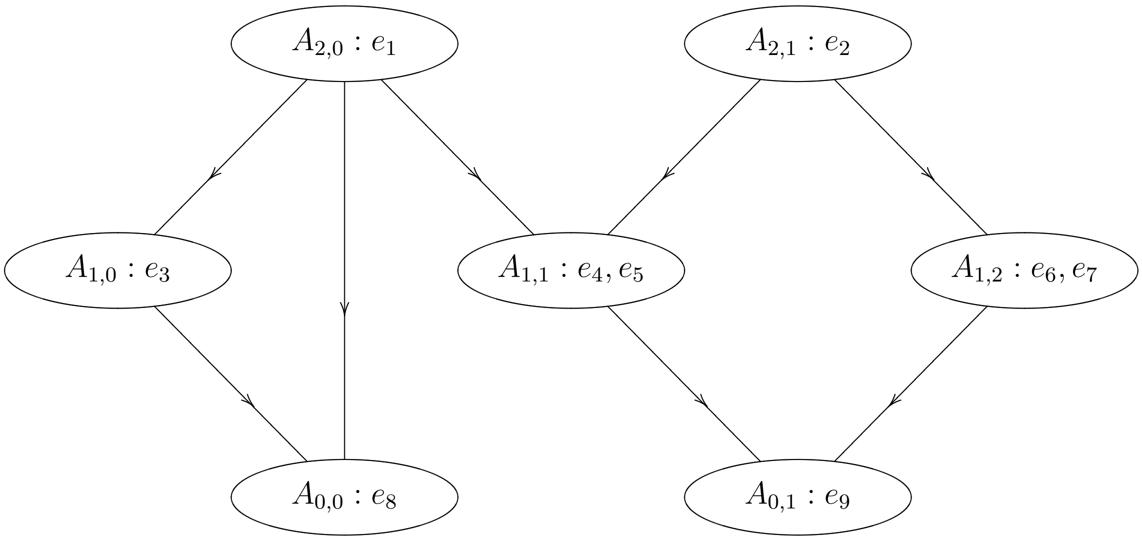
Here, I have written out each $\langle\cdot\rangle_m$ with each generator it contains to more plainly indicate its contents. In fact, each hierarchically simple evolution subalgebra is generated by any single generator it contains (Corollary 1.3.23). So, we see that the hierarchically simple evolution subalgebras are as follows.

$$\begin{aligned}
&A_{0,0} = \langle e_8\rangle && A_{0,1} = \langle e_9\rangle && \\
&A_{1,0} = \langle e_3\rangle_0 && A_{1,1} = \langle e_4, e_5\rangle_0 && A_{1,2} = \langle e_6, e_7\rangle_0 \\
&A_{2,0} = \langle e_1\rangle_1 && A_{2,1} = \langle e_2\rangle_1 &&
\end{aligned}$$

Finally, to construct the skeleton, $\mathbb{S}\langle R,E,I\rangle$, we take each A_{ij} to be a generator and then construct the multiplication table based off of the multiplication table for $\Omega\langle R,E,I\rangle$ according to Definition 1.3.24.

$$\begin{aligned}
&A_{0,1}^2 = A_{0,1} && A_{0,0}^2 = A_{0,0} && \\
&A_{1,0}^2 = A_{0,0} + A_{0,0} && A_{1,1}^2 = A_{0,1} && A_{1,2}^2 = A_{0,1} \\
&A_{2,0}^2 = A_{1,0} + A_{0,0} + A_{1,1} && A_{2,1}^2 = A_{1,1} + A_{1,2} &&
\end{aligned}$$

One can easily represent skeleton graphically, to gain a visual representation of the structure of an evolution algebra. To do this, we simply represent each $A_{i,j}$ as a node. For any two hierarchically simple evolution subalgebras, $A_{i,j}$ and $A_{k,l}$, we draw an arrow from $A_{i,j}$'s node to $A_{k,l}$'s node when $A_{k,l} \prec A_{i,j}^2$. So, for our example, we have the following graphical representation of the skeleton.



This representation immediately reveals which evolution subalgebras are inside which other evolution subalgebras. For instance, we can see that

$$\langle e_9 \rangle \subseteq \langle e_2 \rangle \text{ and } \langle e_1 \rangle \cap \langle e_2 \rangle = \langle e_4, e_5 \rangle$$

Thus, we see that the skeleton of an evolution algebra really is advantageous in studying the structure the evolution algebra. Furthermore, since the skeleton of a skeleton is itself, one may use skeletons to classify evolution algebras. For instance, there are infinitely many evolution algebras with the skeleton just given (up to isomorphism, of course). However, they all share this common structure.

Chapter 2

Relation to Markov Chains

2.1 Definition of Correspondence and Initial Results

2.1.1 Definition of Homogeneous Markov Chain

Definition 2.1.1. Let $X = \{X_0, X_1, X_2, \dots\}$ be a set of random variables over a countable state space S. Then X is a **Markov chain** if it has the **Markov property**:

$$\Pr(X_{n+1} = e_{n+1} | X_n = e_n, X_{n-1} = e_{n-1}, \dots, X_0 = e_0) = \Pr(X_{n+1} = e_{n+1} | X_n = e_n)$$

for all $n \in \mathbb{N}$ and $e_1, \dots, e_{n+1} \in S$.

We can imagine X as some system acquiring different discrete states at discrete time steps, its first state being determined by X_0, its second by X_1, and so on. The Markov property states that which state the system changes to next depends only on its current state and the current time step.

Definition 2.1.2. We say that X is **homogeneous** if

$$\Pr(X_{n+1} = e_i | X_n = e_j) = \Pr(X_n = e_i | X_{n-1} = e_j)$$

for all $n \in \mathbb{Z}^+$ and all $e_i, e_j \in S$.

This means that the current time step of the system does not affect which state it will go to next: the next state only depends on the current state.

2.1.2 Definition of Correspondence

We can form a natural correspondence between evolutionary algebras and Markov chains. From a homogeneous Markov chain X with state space S, if each each element of S has a nonzero probability of transitioning to only finitely many other states, we can form an evolution algebra.

Define the evolution algebra corresponding to X, $\Omega\langle\mathbb{R}, S, I\rangle$, as follows. Note that we are now working over the field of real numbers rather than an arbitrary field. We define the multiplication table like so:

$$e_j^2 = \sum_{e_i \in S}^{\not\infty} \Pr(X_{m+1} = e_i | X_m = e_j) e_i$$

Thus, squaring a state e_j results in a linear combination of states, where the coefficient of each e_i is the probability that a system in state e_j will transition into state e_i.

Notation 2.1.3. For convenience, I will use the following shorthand. Let $p_{ij}^{(m)}$ be defined:

$$p_{ij}^{(m)} = \Pr(X_m = e_i | X_0 = e_j)$$

for $m \in \mathbb{N}$. For $p_{ij}^{(1)}$, I will just write p_{ij}. Since we are working with homogeneous Markov chains, $p_{ij}^{(m)}$ is the probability that if our system begins in state e_j, it will be in state e_i after m steps.

The transition probabilities of a Markov chain can be represented in a matrix P, where $P_{ij} = p_{ji}$. Notice that P is the transpose of the evolution operator for Ω, L. That is,

$$P = \begin{pmatrix} p_{11} & p_{21} & \cdots & p_{n1} & \cdots \\ p_{12} & p_{22} & \cdots & p_{n2} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \\ p_{1n} & p_{2n} & \cdots & p_{nn} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

and

$$L = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} & \cdots \\ p_{21} & p_{22} & \cdots & p_{2n} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} & \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Recall that these matrices will have dimension equal to the cardinality of the state space, and thus can be finite our countably infinite. However, given that each state has a nonzero probability of transitioning to only finitely many other states, each column of L and row of P has only finitely many nonzero entries. To generalize this relation between the evolution operator and transition probabilities, we have the following theorem.

Theorem 2.1.4. *Let X be a Markov chain with state space S and a corresponding evolution algebra $\Omega\langle\mathbb{R}, S, I\rangle$ with evolution operator L. Then L has the following probabilistic significance:*

$$(L^m)_{ij} = \Pr(X_m = e_i | X_0 = e_j)$$

where $e_i \in S$. That is, $\rho_i L^m(e_j) = \Pr(X_m = e_i | X_0 = e_j)$.

Proof. To this end, I will invoke the Chapman-Kolmogorov equation, which states that (for homogeneous Markov chains)

$$\Pr(X_m = e_i | X_0 = e_j) = \sum_{e_k \in S}^{\not\infty} (\Pr(X_{m-1} = e_k | X_0 = e_j) \Pr(X_m = e_i | X_{m-1} = e_k))$$

or in our condensed notation, this says:

$$p_{ij}^{(m)} = \sum_{k=1}^{n} (p_{kj}^{(m-1)} p_{ik})$$

We may proceed to prove the theorem through simple induction. The base case says that $(L^1)_{ij} = p_{ij}^{(1)}$ and follows immediately from the definition of L. So suppose that

$(L^m)_{ij} = p_{ij}^{(m)}$. Then, by the Chapman-Kolmogorov equation, we have:

$$\begin{aligned} L^{m+1} &= L^m L \\ &= \begin{pmatrix} p_{11}^{(m)} & p_{12}^{(m)} & \cdots & p_{1n}^{(m)} & \cdots \\ p_{21}^{(m)} & p_{22}^{(m)} & \cdots & p_{2n}^{(m)} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \\ p_{n1}^{(m)} & p_{n2}^{(m)} & \cdots & p_{nn}^{(m)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \cdot \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} & \cdots \\ p_{21} & p_{22} & \cdots & p_{2n} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \\ &= \begin{pmatrix} \sum_k p_{1k}^{(m)} p_{k1} & \sum_k p_{1k}^{(m)} p_{k2} & \cdots & \sum_k p_{1k}^{(m)} p_{kn} & \cdots \\ \sum_k p_{2k}^{(m)} p_{k1} & \sum_k p_{2k}^{(m)} p_{k2} & \cdots & \sum_k p_{2k}^{(m)} p_{kn} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \\ \sum_k p_{nk}^{(m)} p_{k1} & \sum_k p_{nk}^{(m)} p_{k2} & \cdots & \sum_k p_{nk}^{(m)} p_{kn} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \\ &= \left(p_{ij}^{(m+1)}\right) \end{aligned}$$

Note that although these matrices might be infinite, we can mutliply them since only finitely many elements in each column are nonzero. Thus, we have shown that $(L^m)_{ij} = Pr(X_m = e_i | X_0 = e_j)$ for all m. □

Remark 2.1.5. Note that all evolution algebras derived from some Markov chain have genetic realization. Likewise, any evolution algebra with genetic realization has some corresponding Markov chain.

2.2 Visitation, Destination, and Persistence

2.2.1 Definitions of Visitation and Destination

Definition 2.2.1. Let $\Omega\langle \mathbb{R}, E, I\rangle$ have genetic realization and an evolution operator L. That is, $\Omega\langle \mathbb{R}, E, I\rangle$ corresponds to some homogeneous Markov chain X. The **visitation operator**, $V_j^{(n)}$, tells us the probability that the system will be at state $e_j \in E$ for the first time after n steps. We define it recursively:

$$V_j^{(n)} = \begin{cases} \rho_j L & \text{if } n = 1 \\ V_j^{(n-1)} \rho_j^o L & \text{if } n > 1 \end{cases}$$

Recall that ρ_j and ρ_j^o are the projection and deletion mappings given in Definition 1.3.1. Since L, ρ_j, and ρ_j^o are linear operators, $V_j^{(n)}$ is a linear operator as well. Expanding the recursive definition, we find:

$$V_j^{(n)} = \rho_j L(\rho_j^o L)^{n-1}$$

Lemma 2.2.2. *$V_j^{(n)}(e_i) = Pr(X_n = e_j \cap X_{n-1} \neq e_j \cap \cdots \cap X_1 \neq e_j | X_0 = e_i)e_j$ for all $i, j \in \mathbb{N}$ and $n \in \mathbb{Z}^+$. Note that we do not exclude the possibility $i = j$. Thus, $V_j^{(n)}(e_i)$ is the probability that e_j will appear for the first time after the initial state of the system.*

Proof. Given the recursive definition of the visitation operation, I use induction. For $n = 1$, we have, by Theorem :

$$\begin{aligned} V_j^{(n)}(e_i) &= \rho_j L(e_i) \\ &= p_{ji} e_j \\ &= \Pr(X_1 = e_j | X_0 = e_i) e_j \end{aligned}$$

Suppose that, for all $i, j \in \mathbb{N}$,

$$V_j^{(n)}(e_i) = \Pr(X_n = e_j \cap X_{n-1} \neq e_j \cap \cdots \cap X_1 \neq e_j | X_0 = e_i) e_j$$

for some n. Then we have:

$$\begin{aligned} V_j^{(n+1)}(e_i) &= V^{(n)} \rho_j^o L(e_i) \\ &= V_j^{(n)} \rho_j^o \Big(\sum_{e_k \in E}^{\infty} p_{ki} e_k \Big) \\ &= V_j^{(n)} \Big(\sum_{e_k \in E, k \neq j}^{\infty} p_{ki} e_k \Big) \\ &= \sum_{e_k \in E, k \neq j}^{\infty} p_{ki} V_j^{(n)} e_k \end{aligned}$$

Expanding $V_j^{(n)}$ by the inductive assumption, we have:

$$\begin{aligned}
V_j^{(n+1)}(e_i) &= \sum_{e_k \in E, k \neq j}^{\not\infty} p_{ki} \Pr(X_{n+1} = e_j \cap X_n \neq e_j \cap \cdots \cap X_2 \neq e_j | X_1 = e_k) e_j \\
&= \sum_{e_k \in E, k \neq j}^{\not\infty} \Pr(X_{n+1} = e_j \cap X_n \neq e_j \cap \cdots \cap X_2 \neq e_j \cap X_1 = e_k | X_0 = e_i) e_j \\
&= \Pr(X_{n+1} = e_j \cap X_n \neq e_j \cap \cdots \cap X_2 \neq e_j \cap (\cup_{e_k \in E, k \neq j} X_1 = e_k) | X_0 = e_i) e_j \\
&= \Pr(X_{n+1} = e_j \cap X_n \neq e_j \cap \cdots \cap X_2 \neq e_j \cap X_1 \neq e_j | X_0 = e_i) e_j
\end{aligned}$$

Thus, we find that $V_j^{(n)} e_i$ is the probability that a system starting in state e_i will reach state e_j for the first time after n steps. □

Definition 2.2.3. Let X be a homogeneous Markov chain and let its corresponding evolution algebra $\Omega\langle \mathbb{R}, E, I \rangle$ have an evolution operator L. The **destination operator**, D_j, tells us the probability that the system will enter state e_j at any point after its initial state. We define it:

$$\begin{aligned}
D_j &= \sum_{m=1}^{\infty} V_j^{(m)} \\
&= \sum_{m=1}^{\infty} \rho_j L (\rho_j^o L)^{m-1}
\end{aligned}$$

The reasoning behind this definition is simple enough: Since e_j can be visited for the first time only once, the event corresponding to a $V_j^{(m)}$ is exclusive of the event corresponding to $V_j^{(n)}$ for any $n \neq m$. Hence, to find the probability that there is some m such that the event corresponding to $V_j^{(m)}$ occurs, we sum over the probabilities for all $V_j^{(m)}$. That is, we find the probability that e_j will be visited for the first time.

By the previous lemma, we can see that this is indeed exactly how the definition of D_j works out:

$$D_j(e_i) = \left(\sum_{m=1}^{\infty} Pr(X_n = e_j \cap X_{n-1} \neq e_j \cap \cdots \cap X_1 \neq e_j | X_0 = e_i) \right) e_j$$

2.2.2 Probabilistic Persistence and Transience

Definition 2.2.4. Let X be a Markov chain with state space $E = \{e_1, e_2, \dots\}$. For any $e_j \in S$, let T_j be the random variable defined:

$$T_j = \inf\{n \geq 1 : X_n = e_j | X_0 = e_j\}$$

We say e_j is **probabilistically transient** if and only if

$$\Pr(T_j = \infty) > 0$$

That is, e_j is probablistically transient if there is a nonzero probability that a system beginning in state e_j will never return to e_j. Furthermore, e_j is called **probabilistically persistent** if and only if it is not probabilistically transient.

Intuitively, a state e_j is probabilistically transient if a system beginning in state e_j has a nonzero probability of never returning to e_j. Conversely, e_j is probabilistically persistent if the system is guaranteed to return to it.

Theorem 2.2.5. *Let $\Omega\langle \mathbb{R}, E, I\rangle$ be an evolution algebra with genetic realization. Then in the Markov chain corresponding to $\Omega\langle \mathbb{R}, E, I\rangle$, $e_j \in E$ is probabilistically persistent if and only if*

$$D_j(e_j) = e_j$$

Proof. Suppose that e_j is probabilistically persistent. Then, for the random variable

$$T_j = \inf\{n \geq 1 : X_n = e_j | X_0 = e_j\}$$

we have

$$\Pr(T_j = \infty) = 0$$

Therefore,

$$\left(\sum_{n=1}^{\infty} Pr(X_n = e_j \cap X_{n-1} \neq e_j \cap \cdots \cap X_1 \neq e_j | X_0 = e_j)\right) = 1$$

and so we conclude

$$\begin{aligned} D_j(e_j) &= \left(\sum_{n=1}^{\infty} Pr(X_n = e_j \cap X_{n-1} \neq e_j \cap \cdots \cap X_1 \neq e_j | X_0 = e_j)\right) e_j \\ &= e_j \end{aligned}$$

Now suppose that $D_j(e_j) = e_j$. Then we have

$$\left(\sum_{m=1}^{\infty} Pr(X_n = e_j \cap X_{n-1} \neq e_j \cap \cdots \cap X_1 \neq e_j | X_0 = e_j)\right) = 1$$

and so $\Pr(T_j = \infty) = 0$. Thus, e_j is probabilistically persistent. □

Chapter 3

Relation to Graph Theory

3.1 Definition of Correspondence and Initial Results

3.1.1 Graph Theoretic Definitions

Definition 3.1.1. Let R be a commutative ring with identity. A **weighted, directed graph** (or **weighted digraph**) is defined $G = (V, E, \mathrm{wt})$ where

- V is a countable set of elements called **vertices**. If V is finite, then G is called finite.
- E is a set of ordered pairs of the elements of V called **edges** such that each vertex appears as the first element in only finitely many ordered pairs. If V is infinite, then this restriction on the edges makes G **locally finite**.
- $\mathrm{wt} : V \times V \to R$ is called the **weight function**, and has the property that:

$$\mathrm{wt}(v, w) \neq 0 \text{ if and only if } (v, w) \in E$$

Definition 3.1.2. Let $G = (V, E, \mathrm{wt})$. We have the following specialized variants of a weighted digraph:

- If the image of wt is $\{0, 1\}$, then G is just called a **directed graph**, or **digraph**. In this case, one usually just writes $G = (V, E)$. Furthermore, $\mathrm{wt}(v, w)$ simply indicates whether or not $(v, w) \in E$ and so is not needed to define G.
- If for all $v, w \in V$, $\mathrm{wt}(v, w) = \mathrm{wt}(w, v)$, then G is just called a **weighted, undirected graph**, or simply **weighted graph**.

Definition 3.1.3. Let $G = (V, E, \text{wt})$. Then we have the following further definitions:

- A sequence of vertices $(v_1, v_2, \ldots, v_n)$ is called a **directed path** if for each (v_i, v_{i+1}) with $1 \leq i < n$, $(v_i, v_{i+1}) \in E$. The **weight** or **length** of the path is defined:
$$\sum_{i=1}^{n-1} \text{wt}(v_i, v_{i+1})$$

- A sequence of vertices $(v_1, v_2, \ldots, v_n)$ is called an **undirected path** if for each (v_i, v_{i+1}) with $1 \leq i < n$, $(v_i, v_{i+1}) \in \ E$ or $(v_{i+1}, v_i) \in E$.

- Let $p = (v_1, v_2, \ldots, v_n)$ be a path such that $v_1 = v_n$. Then p is called a **cycle**. If a graph has no such p, the graph is called **acyclic**.

- If for all $v, w \in V$, there exists a directed path starting at v and ending at w, then G is called **strongly connected**.

- If for all $v, w \in V$, there exists a undirected path starting at v and ending at w, then G is called **weakly connected**.

- If G is undirected, then weakly and strongly connected mean the same thing. In this case, we say G is connected.

- Let $G_0 = (V_0, E_0, \text{wt}_0)$, where $V_0 \subset V$, E_0 is a subset of E restricted to V_0, and $\text{wt}_0 : E_0 \to R$ is given by $\text{wt}_0(v, w) = \text{wt}(v, w)$ for $v, w \in V_0$ if $(v, w) \in E_0$. Then G_0 is a **subgraph** of G. If, for all $v, w \in V_0$, $\text{wt}_0(v, w) = \text{wt}(v, w)$ in general, then G_0 is called an **induced subgraph**.

Definition 3.1.4. Let $G_1 = (V_1, E_1, \text{wt}_1)$ and $G_2(V_2, E_2, \text{wt}_2)$ be weighted digraphs. Let $\phi : V_1 \to V_2$ be a bijection such that for all $v, w \in V_1$:

$$\text{wt}_1(v, w) = \text{wt}_2(\phi(v), \phi(w))$$

Then ϕ is called an **graph isomorphism** and G_1 and G_2 are called **isomorphic**.

3.1.2 Definition of Correspondence

Definition 3.1.5. Let $G = (V, E, \text{wt})$ be a weighted digraph. I define the evolution algebra, $\Omega(G) = \Omega\langle R, V, I\rangle$, corresponding to G, such that I is generated by:

$$\{v_i^2 - \sum_{v_j \in V}^{\not\infty} \text{wt}(v_i, v_j) v_j | v_i \in V\}$$

Note that for each $v_i \in V$, there are only finitely many $v_j \in V$ for which $\mathrm{wt}(v_i, v_j) \neq 0$.

Similarly, given any evolution algebra $\Omega\langle R, V, I\rangle$ such that for each $v_i \in V$, there are only finitely many $v_j \in V$ such that $\rho_j(v_i^2) \neq 0$, we can define a corresponding weighted digraph $G = (V, E, \mathrm{wt})$ such that:

$$\mathrm{wt}(v_i, v_j)v_j = \rho_j(v_i^2)$$

for all $v_i, v_j \in V$. Then $E = \{(v_i, v_j) \in V \times V | \, \mathrm{wt}(v_i, v_j) \neq 0\}$.

Theorem 3.1.6. *Let $G = (V, E, \mathrm{wt}_G)$ and $H = (W, F, \mathrm{wt}_H)$ be weighted digraphs. If G and H are isomorphic, then $\Omega(G) = \Omega\langle R, V, I\rangle$ and $\Omega(H) = \Omega\langle R, W, J\rangle$ are isomorphic.*

Proof. Assume that G and H are isomorphic. Let $\phi : G \to H$ be an isomorphism. We index the elements of V and W such that $\phi : v_i \mapsto w_i$ for all $v_i \in V$ and $w_i \in W$. We can then define an isomorphism ϕ' between $\Omega(G)$ and $\Omega(H)$ as follows. Let $\sum_{v_i \in V}^{\not\infty} a_i v_i$ be an arbitrary element in $\Omega(G)$ with $a_i \in R$:

$$\phi'(\sum_{v_i \in V_1}^{\not\infty} a_i v_i) = \sum_{v_i \in V_1}^{\not\infty} a_i \phi(v_i) = \sum_{w_i \in W}^{\not\infty} a_i w_i$$

To see that ϕ' is in fact an isomorphism, first note that $\phi'^{-1} : W \to V$ is naturally defined in terms of ϕ^{-1}:

$$\phi'^{-1}(\sum_{w_i \in W}^{\not\infty} a_i w_i) = \sum_{w_i \in W}^{\not\infty} a_i \phi^{-1}(w_i) = \sum_{v_i \in V}^{\not\infty} a_i v_i$$

Hence, ϕ' is bijective. To see that it is a homomorphism, we check if ϕ' preserves multiplication between generators. For any $v_i \in V$, we have

$$v_i^2 = \sum_{v_j \in V}^{\not\infty} \mathrm{wt}_G(v_i, v_j)v_j$$

Since G and H are isomorphic, $\mathrm{wt}_G(v_i, v_j) = \mathrm{wt}_H(w_i, w_j)$, and hence

$$w_i^2 = \sum_{w_j \in W}^{\not\infty} \mathrm{wt}_H(w_i, w_j)w_j = \sum_{w_j \in W}^{\not\infty} \mathrm{wt}_G(v_i, v_j)w_j$$

So then:

$$\begin{aligned}\phi'(v_i^2) &= \phi(\sum_{v_j \in V}^{\not\infty} \mathrm{wt}_G(v_i, v_j) v_j) \\ &= \sum_{w_j \in W}^{\not\infty} \mathrm{wt}_G(v_i, w_j) w_j \\ &= w_i^2 \\ &= \phi'(v_i)^2\end{aligned}$$

Hence, if G and H are isomorphic, then $\Omega(G)$ and $\Omega(H)$ are isomorphic. □

3.1.3 Connectivity

Lemma 3.1.7. *Let $G = (V, E, \mathrm{wt})$ be a weighted digraph and $\Omega(G) = \Omega\langle R, V, I\rangle$. If for any distinct $v, w \in V$, w is a descendant of v, then there exists a directed path from v to w.*

Proof. Since w is a descendant of v, there must exist some n such that $w \prec v^{[n]}$. To prove this lemma, I will use induction on n.

Suppose $n = 1$. Then $w \prec v^{[1]}$ and so $(v, w) \in E$. Hence the path (v, w) starts v and ends at w.

Now suppose that $w \prec v^{[n+1]}$ and that for any $w' \in V$, if $w' \prec v^{[n]}$, then there exists a directed path from v to w'. Then there is some $v_n \in V$ such that $v_n \prec v^{[n]}$ and $w \prec v_n^{[1]}$. Thus, there is a directed path, $(v = v_1, v_2, \ldots, v_n)$. But then

$$(v = v_1, v_2, \ldots, v_n, w)$$

is a directed path as well, since $(v_n, w) \in E$.

Therefore, by induction, if for any distinct $v, w \in V$, w is a descendant of v, then there exists a directed path from v to w. □

Lemma 3.1.8. *Let $G = (V, E, \mathrm{wt})$ be a weighted digraph and $\Omega(G) = \Omega\langle R, V, I\rangle$. Then for any $v, w \in V$, $\langle w\rangle \subseteq \langle v\rangle$ if and only if there exists a directed path from v to w.*

Proof. Suppose that $\langle w\rangle \subseteq \langle v\rangle$. Then by Theorem 1.3.5 in Chapter 1, there must be some sequence of descendants, $(w = v_n, v_{n-1}, \ldots, v_1 = v)$. By the previous lemma, there exists a directed path from each v_i to v_{i+1}. Note that in the sequence of descendants given here, v_{i+1} is a descendant of v_i. By linking these paths together, we form a path from v to w.

Now suppose that exists a directed path from w to v. Then there exists a corresponding sequence of descendants from v to w. By the transitivity of the subset relation, $\langle v\rangle$ contains each of the subalgebras generated by the elements in this sequence of descendants. Therefore, $\langle w\rangle \subseteq \langle v\rangle$. □

Corollary 3.1.9. *Let $G = (V, E, \mathrm{wt})$ be a weighted digraph and $\Omega(G) = \Omega\langle R, V, I\rangle$. Then for any distinct $v, w \in V$, if there exists no undirected path between v and w, then $\langle v\rangle \cap \langle w\rangle = \emptyset$.*

Proof. Suppose there is some $v_i \in \langle v\rangle \cap \langle w\rangle$. Then $\langle v_i\rangle \subseteq \langle v\rangle$ and $\langle v_i\rangle \subseteq \langle w\rangle$. Thus, there is a directed path from v to v_i and from w to v_i. But then there is an undirected path connecting v and w (namely, the one containing v_i). However, I assumed that no such path existed. Hence, no such v_i exists. □

Theorem 3.1.10. *Let $G = (V, E, \mathrm{wt})$ be a weighted digraph.*

1. *G is weakly connected if and only if $\Omega(G)$ is connected in the algebraic sense.*

2. *G is strongly connected if and only if $\Omega(G)$ is simple.*

Proof. 1. Suppose G is weakly connected. Let A_0 be an evolution subalgebra with generators $V_0 \subseteq V$. It suffices to show that $\langle V\backslash V_0\rangle \cap A_0 \neq \emptyset$. Let $v \in V\backslash V_0$ and $w \in V_0$. Then there exists an undirected path $(v = v_1, v_2, \ldots, v_n = w)$ in G. Along this path, there must be some $1 \leq i \leq n$ such that $v_i \in V\backslash V_0$ and $v_{i+1} \in V_0$. Then $\langle v_i\rangle \cap \langle v_{i+1}\rangle \neq \emptyset$. Since $\langle v_i\rangle \subseteq \langle V\backslash V_0\rangle$ and $\langle v\rangle \subseteq \langle V_0\rangle = A$, then $\langle V\backslash V_0\rangle \cap A_0 \neq \emptyset$. This means that $\Omega(G)$ cannot be decomposed into a direct sum of evolution algebras and so $\Omega(G)$ is connected.

Now suppose that G is not weakly connected. That is, there exist some $v, w \in V$ such that no undirected path exists starting with v and ending with w. Let W be the set of all generators for which w is connected via an undirected path. Then note that for any $w_i \in W$ and $v_i \in V\backslash W$, there exists no undirected path connecting w_i and v_i. By the previous corollary then, $\langle v_i\rangle \cap \langle w_i\rangle = \emptyset$. Therefore, $\langle W\rangle \cap \langle V\backslash W\rangle = \emptyset$. Hence

$$\Omega(G) = \langle W\rangle \oplus \langle V\backslash W\rangle$$

and so $\Omega(G)$ is not connected.

2. By Lemma 3.1.8, for any $v, w \in V$, there exists a path starting v and ending at w and another path starting at path w and ending at v if and only if $\langle v \rangle = \langle w \rangle$. Thus, G is strongly connected if and only if $\Omega(G)$ is simple.

□

Corollary 3.1.11. *Let G be a weighted graph. Then $\Omega(G)$ is connected if and only if it is simple.*

3.1.4 Cycles and Trees

Definition 3.1.12. Let $G = (V, E, \mathrm{wt})$.

1. If there exists some $v \in V$ such that for all $w \in V$, there is a directed path from v to w, and G is acyclic, we call G a **tree**. In this case, v is called the **root** of G.

2. If G has a subgraph $T = (V, E_0, \mathrm{wt}_0)$ such that T is a tree, then we call T a **spanning tree** of G.

Theorem 3.1.13. *Let $G = (V, E, \mathrm{wt})$. For any $v \in V$, G has a spanning tree with v as its root if and only if $\langle v \rangle = \Omega(G)$.*

Proof. This follows immediately from Lemma 3.1.8. □

Lemma 3.1.14. *Let $G = (V, E, \mathrm{wt})$. Then G has a cycle if and only if there exist some distinct $v, w \in V$ such that $\langle v \rangle = \langle w \rangle$ in $\Omega(G)$.*

Proof. Suppose that G has a cycle c beginning and ending at $v \in V$ that contains $w \in V$ such that $v \neq w$. Then c has a directed subpath beginning at v and ending at w, and c has a directed subpath beginning at w and ending at v. Hence, by Lemma 3.1.8, $\langle v \rangle = \langle w \rangle$.

Now suppose that G has two distinct vertices $v, w \in V$ such that $\langle v \rangle = \langle w \rangle$. Then there exists a directed path from v to w and another directed path from w to v. By concatenating these two paths, we find a path from v to v. Hence, G has a cycle. □

Theorem 3.1.15. *Let $G = (V, E, \mathrm{wt})$. Then G is a tree if and only if the following conditions hold for $\Omega(G)$:*

1. *There exists some $v \in V$ such that $\langle v \rangle = \Omega(G)$.*

2. *No distinct $v, w \in V$ exist such that $\langle v \rangle = \langle w \rangle$ in $\Omega(G)$.*

Proof. This follows immediately from the previous lemma and theorem. □

3.1.5 Path Lengths

Theorem 3.1.16. *Let $G = (V, E)$ be a digraph and $\Omega(G) = \Omega\langle \mathbb{Z}, E, I\rangle$ its corresponding evolution algebra with an evolution operator L. Let $v_i, v_j \in V$ be such that $\rho_j L^n(v_i) = p_n v_j$ for $n > 0$. Then p_n is number of paths beginning at v_i and ending at v_j of length n.*

Proof. To this end, I use induction. First, take $n = 1$. Of course, there can only be a single path from v_i to v_j; namely, the path (v_i, v_j). This path exists if and only if $(v_i, v_j) \in E$. So we have:

$$\rho_j L^1(v_i) = \rho_j(v_i^2) = \begin{cases} 1v_j & \text{if } (v_i, v_j) \in E \\ 0 & \text{if } (v_i, v_j) \notin E \end{cases}$$

Thus, the theorem works for $n = 1$.

Fix $n > 0$. Then for all $v_i, v_j \in V$, suppose that for $\rho_j L^n(v_i) = p_n v_j$, p_n is the number of paths of length n from v_i to v_j. Then I must show that for $\rho_j L^{n+1}(v_i) = p_{n+1} v_j$, p_{n+1} is the number of paths of length $n + 1$ from v_i to v_j. So we have:

$$\begin{aligned} \rho_j L^{n+1}(v_i) &= \rho_j L^n\Big(\sum_{v_k \in V}^{\not\infty} a_{ik} v_k\Big) \\ &= \sum_{v_k \in V}^{\not\infty} a_{ik} \rho_j L^n(v_k) \end{aligned}$$

Note that $a_{ik} = 1$ if $(v_i, v_k) \in E$ and 0 otherwise. By assumption, each of the $\rho_j L^n(v_k)$ is the number of paths of length n from v_k to v_j. The a_{ik} guarantees that the only v_k for which this is calculated are the ones that v_i has an edge to; hence, these v_k are exactly the vertices that may be the second vertex in a path starting at v_i. Thus,

$$\rho_j L^{n+1}(v_i) = \sum_{v_k \in V}^{\not\infty} a_{ik} \rho_j L^n(v_k) = p_{n+1}$$

is in fact the number of paths from v_i to v_j of length $n + 1$.

Therefore, by induction, $\rho_j L^n(v_i) = p_n v_j$. □

Corollary 3.1.17. *Let $G = (V, E)$ be a digraph and $\Omega(G) = \Omega\langle \mathbb{Z}, E, I\rangle$ its corresponding evolution algebra with an evolution operator L. Let $v_i, v_j \in V$ such that there exists a path from v_i to v_j. Then the shortest directed path from v_i to v_j is given* *by:*

$$\min(\{k|\rho_j L^k(v_i) \neq 0\})$$

3.2 Graph Theoretic Problems in the Language of Evolution Algebras

3.2.1 k-Coloring Problem

Definition 3.2.1. Let $G = (V, E)$ be an unweighted, undirected graph. Let $C = \{1, \ldots, k\}$ and $\xi : V \to C$. Then ξ is called a k**-coloring** of G if for all $(v, w) \in E$, $\xi(v) \neq \xi(w)$. In this case, G is called k**-colorable**. In this case, $\Omega(G)$ is also called k-colorable.

The k-coloring problem is simply the problem of determining whether an arbitrary (unweighted, undirected) graph G is k-colorable. In general, this problem is very difficult to solve; for $k > 2$, the only method we know of determining whether or not an arbitrary graph is k-colorable is by an exhaustive search (or something equivalently time-consuming). However, when $k = 2$, we have a simple solution. It turns out that a graph is 2-colorable if and only if it has no cycles of odd length. For the purpose of demonstrating how one would use the theory of evolution algebras to solve graph theoretic problems, I will derive this result using evolution algebras.

Definition 3.2.2. Let $G = (V, E)$ be a graph. G is called a **bipartite graph** if V may be divided into two proper subsets, V_0 and V_1, such that for any $v, w \in V_i$, $(v, w) \notin E$. Similarly, I will call $\Omega(G)$ a **bipartite evolution algebra**. That is, an evolution algebra $\Omega\langle\mathbb{Z}, V, I\rangle$ is bipartite if there exist disjoint $V_0, V_1 \subsetneq V$ such that $V_0 \cup V_1 = V$ and for all $v \in V_0$, v^2 is a linear combination of elements in V_1 and vice versa.

Of course, the property of being bipartite is the same as being 2-colorable in disguise:

Lemma 3.2.3. *Let $\Omega\langle\mathbb{Z}, V, I\rangle$ correspond to some graph. Let $C = \{0, 1\}$. Then there exists a mapping $\xi : V \to C$ such that for any $v, w \in V$, if $v \prec w^2$, then $\xi(v) \neq \xi(w)$ if and only if $\Omega\langle\mathbb{Z}, V, I\rangle$ is bipartite.*

Proof. Suppose such a mapping ξ exists for $\Omega\langle\mathbb{Z}, V, I\rangle$. Then let $V_0 \subseteq V$ be the inverse image $\xi^{-1}(0)$ and let $V_1 \subseteq V$ of $\xi(1)$. Then V_0 and V_1 are certainly disjoint and $V_0 \cup V_1 = V$. Furthermore, for any $v \in V_0$ and $w \in V$, since $\xi(v) = 0$, if $w \prec v^2$, then $\xi(w) = 1$ and therefore $w \in V_1$. Hence, v^2 is a linear combination of elements in V_1. The same holds for any $v \in V_1$ as well of course.

Now suppose that $\Omega\langle \mathbb{Z}, V, I\rangle$ is bipartite. Let V_0 and V_1 be the two parts of V. Let $\xi : V \to C$ be defined:

$$\xi(v) = \begin{cases} 0 & \text{if } v \in V_0 \\ 1 & \text{if } v \in V_1 \end{cases}$$

Of course, it follows that for any $v \in V_0$, if $w \prec v^2$, $w \in V_1$, and vice versa. □

Lemma 3.2.4. *Let $\Omega\langle \mathbb{Z}, V, I\rangle$ correspond to some (unweighted, undirected) graph. Let L be its evolution operator. Then $\Omega\langle \mathbb{Z}, V, I\rangle$ is bipartite if and only if for all $v_i \in V$ and odd $k \in \mathbb{Z}^+$, $\rho_i L^k(v_i) = 0$. Recall that $\rho_i L^k(v_i)$ is number of cycles containing v_i of length k, as established in the previous section.*

Proof. Let $\Omega\langle \mathbb{Z}, V, I\rangle$ be bipartite. Let $V_0, V_1 \subsetneq V$ be such that $V_0 \cup V_1 = V$ and for any $v \in V_0$, $v^2 = L(v)$ is a linear combination of elements in V_1 (and vice versa). Let $\sum_{v_i \in V_0}^{\not\infty} a_i v_i$ be an arbitrary linear combination of elements in V_0. Then we have:

$$L(\sum_{v_i \in V_0}^{\not\infty} a_i v_i) = \sum_{v_i \in V_0}^{\not\infty} a_i L(v_i)$$

Thus, $L(\sum_{v_i \in V_0}^{\not\infty} a_i v_i)$ is a linear combination of elements of V_1. Of course, the converse holds for V_1 as well. Hence, for any $v_i \in V$ and odd k, $L^k(v_i)$ will be a linear combination of elements in V_1. Hence, $\rho_i L^k(v_i) = 0$. Again, the same holds for V_1 as well.

Now, assume that for all $v_i \in V$ and odd $k \in \mathbb{Z}^+$, $\rho_i L^k(v_i) = 0$. It suffices to show that if $\Omega\langle \mathbb{Z}, V, I\rangle$ is connected, then it is bipartite. Since $\Omega\langle \mathbb{Z}, V, I\rangle$ is connected, it must be simple, and therefore, for any $v \in V$, $\langle v\rangle = \Omega\langle \mathbb{Z}, V, I\rangle$. Fix a v. Let $V_0 = \{w \in V | k \in \mathbb{N}, w \prec L^{2k}(v)\}$ and let $V_1 = \{w \in V | k \in \mathbb{N}, w \prec L^{2k+1}(v)\}$. Since $\Omega\langle \mathbb{Z}, V, I\rangle$ is nonnegative, for any $w \in V$, $w \in \langle v\rangle$ if and only if w is a descendant of v. Hence, $V_0 \cup V_1 = \langle v\rangle \cap V = V$. Furthermore, by the definition of V_0 and V_1, for any $w \in V_0$, $L(w)$ is a linear combination of elements in V_1 and vice versa.

We need to make sure that V_0 and V_1 are disjoint, however. So let $w \in V_0 \cap V_1$. Then there exists some odd k and even l such that $w \prec L^k(v)$ and $w \prec L^l(v)$. Since $\Omega\langle \mathbb{Z}, V, I\rangle$ corresponds to an undirected graph, this mean that $v \prec L^l(w)$. But then

$v \prec L^{k+l}(v)$ and $k+l$ is odd. This contradicts our initial assumption that for all $v_i \in V$ and odd $j \in \mathbb{Z}^+$, $\rho_i L^j(v_i) = 0$. So no such w can exist. Therefore, $\Omega\langle \mathbb{Z}, V, I\rangle$ is bipartite. □

Theorem 3.2.5. *Let $\Omega\langle \mathbb{Z}, V, I\rangle$ correspond to some graph. Let $C = \{0, 1\}$. Then there exists a mapping $\xi : V \to C$ such that for any $v, w \in V$, if $v \prec w^2$, then $\xi(v) \neq \xi(w)$ if and only if for any $v_i \in V$, $\rho_i L^k(v) = 0$ if k is odd.*

Proof. This follows directly from the previous two lemmas. □

3.3 A Very Brief Section on Hierarchies of Graphs

Proposition 3.3.1. *Let $G = (V, E, \mathrm{wt})$ be a weighted digraph and $\Omega(G) = \Omega\langle R, V, I\rangle$ its corresponding evolution algebra. Let $V_{i,j} \subseteq V$ be the generators contained in the hierarchically simple evolution subalgebra $A_{i,j} \subseteq \Omega(G)$. Then for the subgraph $G_{i,j} \subseteq G$ induced by $V_{i,j}$, $A_{i,j} = \Omega(G_{i,j})$ under the modified operator $\overset{i}{\cdot}$.*

Proof. Let v_k be a generator in $A_{i,j}$. Then $v_k \overset{i}{\cdot} v_k = \rho_{A_{i,j}}(v_k^2)$. That is, for any other generator $v_l \in A_{i,j}$,

$$\rho_l v_k \overset{i}{\cdot} v_k = \rho_l v_k^2 = \mathrm{wt}(v_k, v_l)$$

Since $G_{i,j}$ is an induced subgraph and contains both v_l and v_k as vertices, the edge (v_k, v_l) both exists and has the same weight in $G_{i,j}$. Therefore, $A_{i,j} = \Omega(G_{i,j})$ under the modified operator $\overset{i}{\cdot}$. □

Corollary 3.3.2. *Given the premises of the previous problem, $G_{i,j}$ is strongly connected.*

Proof. This follows directly from the fact that $A_{i,j}$ is simple. □

Proposition 3.3.3. *Let $G = (V, E, \mathrm{wt})$. If G is acyclic, then each hierarchically simple subalgebra of $\Omega(G)$ contains only one generator.*

Proof. This follows directly from the fact that each hierarchically simple subalgebra of $\Omega(G)$ is, in fact, simple, and therefore for all generators v, w in a particular hierarchically simple subalgebra, $\langle v\rangle = \langle w\rangle$. Thus, there must be directed paths from v to w and vice versa. If $v \neq w$, this would imply the existence of a loop in G, and therefore $v = w$. □

Chapter 4

Relation to Formal Grammars

4.1 An Introduction to Formal Grammars and Decidability

4.1.1 Definition of Formal Grammar

Notation 4.1.1. Let S be a set. Then S^* is the set of all finite sequences of the elements of S, including the empty sequence.

Definition 4.1.2. Let Σ be a finite set. Then a **formal language** L is a subset of Σ^*. We call Σ the **alphabet** of L and we call the elements of Σ **symbols**. We call each element of Σ^* a **string (of symbols)**.

Formal grammars are useful ways of defining formal languages. A formal grammar consists of rules by which strings of symbols can be generated. Thus, a formal grammar G defines a formal language L insofar as a string of symbols is in L if and only if it can be generated using the rules given by G.

Definition 4.1.3. An **(unrestricted) formal grammar** $G = (N, \Sigma, P, S)$, with N, Σ, P, and S defined as follows.

- N is a finite set of symbols. We call these **nonterminal symbols**.
- Σ is the alphabet of the formal language. Σ and N must be disjoint. Symbols in the alphabet are called **terminal symbols**.
- P is a finite set of **production rules**, each of the form

$$(\Sigma \cup N)^* N (\Sigma \cup N)^* \to (\Sigma \cup N).$$

That is, the left hand side of each rule consists of any finite sequence of terminal and nonterminal symbols, followed by a nonterminal symbol, followed by another finite sequence of terminal and nonterminal symbols. The right hand side consists of a finite sequence of terminal and nonterminal symbols.

- $S \in N$ is the starting symbol used in the production of strings.

A formal grammar $G = (N, \Sigma, P, S)$, then, describes a formal language L as follows. A string s is in L if and only if $s \in \Sigma^*$ (s consists entirely of terminal symbols) and s can be produced from the symbol S using a finite sequence of production rules.

We can best understand how to use formal grammars through example:

Example. Let $G = (N, \Sigma, P, S)$ where $N = \{S, T, U\}$, $\Sigma = \{a, b, c, d\}$, and P consists of the following rules:

1. $S \to TU$
2. $T \to aTb$
3. $T \to \varepsilon$
4. $bU \to Uc$
5. $U \to \varepsilon$

where ε is the empty string. We may produce strings in the formal language by beginning with S, and substituting it using the production rules:

$$\begin{aligned} S &\to TU && \text{by rule 1} \\ &\to aTbU && \text{by rule 2} \\ &\to aaaaaTbbbbbU && \text{by rule 2 applied four more times} \\ &\to aaaaaTbbbUcc && \text{by rule 4 applied twice} \\ &\to aaaaabbbcc && \text{by rules 3 and 5} \end{aligned}$$

Thus, $aaaaabbbcc$ is in the formal language described by G. In fact, the formal language corresponding to G includes exactly the strings that contain m 'a's, n 'b's, and p 'c's such that $m - n = p$. Thus, this particular formal grammar encodes substraction with the natural numbers!

4.1.2 Effective Method and Decidability

Definition 4.1.4. Given a (finite) description[1], a domain, and a subset of the domain, a **decision problem** is the problem of determing if an arbitrary element from the domain is in that subset. For instance, if the set is a formal language, the finite description of the set may be the formal grammar that generates that language. The decision problem is the problem of determining if an arbitrary string of symbols can be generated by that formal grammar.

Given a decision problem, a **method** is some (finite) description of steps that, when carried out on an element in the domain, can result in true, result in false, or never terminate (thus not answering the problem). Typically, a method is an algorithm of some sort.

Consider some decision problem concerning the contents of a set S. We say that a method of solving this problem is an **effective method** if

- For any element, x, in the domain, if $x \in S$, then the method will result in true.
- For any element, x, in the domain, if $x \notin S$, then the method will result in false.
- The method will always terminate after a finite number of steps.

Although these definitions are fairly abstract, decision problems and effective methods are fairly intuitive notions. A method is simply a strategy for solving a problem; it might work, it might not. An effective method is a strategy that will always give the right answer in a finite amount of time.

Example. Let $\Omega\langle R, E, I\rangle$ be an evolution algebra such that E is finite. Let E be our domain. Let $e \in E$ and

$$E_0 = \{f \in E | \exists k \in \mathbb{N}, f \prec e^{[k]}\}$$

Thus, E_0 is the set of all descendants of e, with the above statement being our finite description of the set. Thus, determining if, for any $f \in E$, $f \in E_0$ is a decision problem. Then an effective method for solving this problem would be as follows:

1. If $f = e$, return true.
2. Let $E_1 = \{e\}$ and let $k = 1$.

[1] For instance, a finite description may be a set comprehension statement, like $\{e \in E | e \prec e^2\}$, or it may be a formal grammar, if the set we wish to talk about is a formal language. Often, people will simply use descriptions of the set written in natural language, such as "The set of all prime numbers". It is usual understood that the description of the set could be formalized if necessary.

3. Let $E_2 = \{e_i \in E : e_i \prec e^{[k]}\}$.

4. If $f \in E_2$, return true. If $E_2 \subseteq E_1$, return false. Otherwise, add the contents of E_2 to E_1, add one to k, and go back to the previous step.

Given any $f \in E$, this method will accurately determine whether f is a descendant of e in a finite number of steps. Note that this method would not be effective if E were infinite: for example, if $E = \{e_1, e_2, e_3, \dots\}$ where $e_i^2 = e_{i+1}$, when $e = e_2$ and $f = e_1$, the method would never terminate.

Definition 4.1.5. We say that a decision problem is **decidable** if there exists an effective method for solving it; otherwise, we call it **undecidable**. While many decisions problems are decidable many are not, as demonstrated in the above example. For example, let G be a formal grammar with alphabet Σ, describing a lanuage L. Then it is undecidable if $L = \Sigma^*$. That is, no effective method exists for determining if an arbitrary formal grammar will generate every possible string in Σ^*. Even the problem of determining if an arbitrary element of Σ^* is in L is undecidable.[2]

4.2 Correspondence and Decidability of Evolution Algebras

4.2.1 Definition of Correspondence

The production rules of a formal grammar impose a kind of structure on the set $(\Sigma \cup N)^*$. We can mirror this structure using an evolution algebra. Given a formal grammar $G = (N, \Sigma, P, S)$, we construct an evolution algebra $\Omega\langle \mathbb{Z}, E, I\rangle$ as follows:

Let $E = (\Sigma \cup N)^*$. Multiplication is defined as follows for $e_i \in E$:

$$e_i^2 = \begin{cases} \displaystyle\sum_{e_j \in E_i}^{\infty} e_j & \text{if } e_i \text{ contains nonterminal symbols} \\ 0 & \text{if } e_i \text{ contains only terminal symbols} \end{cases}$$

where $E_i \subseteq E$ is the set of all strings e_j for which there exists a production rule in P that turns e_i into e_j.

[2] For a more technical discussion of formal grammars and decidability, see Sipser [1997]. While I present decidability in terms of effective methods, he presents it in terms of abstract computation devices, such as Turing machines. By the Church-Turing Thesis, these presentations are equivalent. The discussion in Sipser [1997] has significantly more depth than what I can present in a single chapter.

Notation 4.2.1. For a given formal grammar $G = (N, \Sigma, P, S)$ and corresponding evolution algebra $\Omega\langle R, E, I\rangle$, if $a_0a_1 \dots a_n \in (\Sigma \cup N)^*$ then I will write $\overline{a_0a_1 \dots a_n}$ to signify the corresponding element in the algebra.

Example. Returning to the example from the beginning of the chapter, let $G = (N, \Sigma, P, S)$ where $N = \{S, T, U\}$, $\Sigma = \{a, b, c, d\}$, and P consists of the following rules:

1. $S \to TU$
2. $T \to aTb$
3. $T \to \varepsilon$
4. $bU \to Uc$
5. $U \to \varepsilon$

For the corresponding evolution algebra, then, we have the following examples of multiplication:

$$\begin{aligned} \overline{S}^2 &= \overline{TU} \\ \overline{TU}^2 &= \overline{aTbU} + \overline{U} + \overline{T} \\ (\overline{aTbU} + \overline{T} + \overline{U})^2 &= \overline{aaTbbU} + \overline{abU} + \overline{aTUc} + 2\overline{aTb} \\ &\vdots \end{aligned}$$

Theorem 4.2.2. *Let $G = (N, \Sigma, P, S)$ be a formal grammar and let $\Omega\langle \mathbb{Z}, E, I\rangle$ be the corresponding evolution algebra. Then for any $e_i, e_j \in E$, $e_i \prec e_j^{[n]}$ if and only if there exists a sequence of n production rules that turns e_j into e_i.*

Proof. To this end, I will use induction over n. For $n = 0$, $e_i \prec e_j^{[0]}$ if and only if $e_i = e_j$. Of course, a sequence of zero production rules will only turn a string into itself, so the theorem holds for $n = 0$.

Now, suppose that for any $e_k, e_l \in E$ and for some fixed n, $e_k \prec e_l^{[n]}$ if and only if there exists a sequence of n production rules that will turn e_k into e_l. Of course, e_j can turn into e_i in $n + 1$ steps if and only if there is some e_k that occurs in e_j^2 such that e_k can turn into e_i in n steps. By our assumption, this can happen if and only if $e_i \prec e_j^{[n+1]}$.

Therefore, by induction, for any $e_i, e_j \in E$ and $n \in \mathbb{N}$, $e_i \prec e_j^{[n]}$ if and only if there exists a sequence of n production rules that will turn e_j into e_i. □

Corollary 4.2.3. *Let $G = (N, \Sigma, P, S)$ be a formal grammar and L the language it defines. Let $\Omega\langle\mathbb{Z}, E, I\rangle$ be the corresponding evolution algebra. Then for any string $x \in \Sigma^*$, $x \in L$ if and only if $\overline{x}$ is a descendant of $\overline{S}$ in $\Omega\langle\mathbb{Z}, E, I\rangle$.*

4.2.2 Evolution Algebras and Decidability

Theorem 4.2.4. *Let $\Omega\langle R, E, I\rangle$ be an evolution algebra such that E is infinite. Let $e \in E$ and let D be the set of descendants of e. Then the decision problem of determining if an element in E is in D is undecidable in general.*

Proof. Let $G = (N, \Sigma, P, S)$ be a formal grammar describing the language L, and let $\Omega\langle\mathbb{Z}, E, I\rangle$ be the corresponding evolution algebra. Let D be set of descendants of $\overline{S} \in E$. Consider the decision problem of determining if an element of E is contained in D.

Suppose that there exists an effective method for solving this decision problem for an arbitrary element of E is in D. Let $x \in \Sigma^*$. Then $\overline{x} \in D$ if and only if $x \in L$. By assumption, we have an effective method of determining if $\overline{x} \in D$. But then we have an effective method of determining if $x \in L$. As noted in the section on effective methods and decidability, no such effective method can exist. Therefore, no effective method exists for determining if an element of E is in D. Hence, the problem is undecidable in general. □

Corollary 4.2.5. *Let $\Gamma\langle R, E, I\rangle$ be a gametic algebra such that E is infinite. Let $e_i, e_j, e_k \in E$. Then the problem of determining if there exists some l such that $e_k \prec (e_i e_j)^{[l]}$ is undecidable in general.*

Proof. This follows directly from the fact that evolution algebras are gametic algebras. □

Remark 4.2.6. Evolution algebra theory is largely concerned with determining when one generator descends from another generator. Indeed, in the underlying biology, this problem corresponds to determining if one species (or piece of genetic material) can evolve from another species (or piece of genetic material). For evolution algebras corresponding to Markov chains, this corresponds to determining whether one state may arise from another. So this last theorem may dampen the hearts of those who hoped that, given the proper knowledge of genetics and computational power, we would be able to determine the possible paths of evolution for a population (assuming that evolution and gametic algebras are a somewhat accurate abstraction of population genetics).

Of course, just as there are many formal grammars that do have decidable contents, there are many evolution algebras that have decidable descendant relations. The example given in the section on effective methods and decidability shows that any evolution algebra with a finite basis does have a decidable descendant relation. It may be a worthwhile effort to categorize evolution algebras based on their decidability properties, just as theoretical computer scientists have done with grammars. The properties that separate different formal grammars may have algebraic analogues.

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